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DISTRIBUTION OF THE NUMBER OF TIMES M/M/2/N QUEUING SYSTEM WITH HETEROGENEOUS SERVERS REACHES ITS CAPACITY IN TIME T SUBJECT TO CATASTROPHES AND RESTORATIONS

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ABSTRACT

In this paper, we study the distribution of the number of times that a limited capacity two heterogeneous servers Markovian queue subject to catastrophes and restorations reaches its capacity in time t. The occurrence of a catastrophe makes the system empty instantly but the system takes its own time to be ready to accept new customers, this time is referred to as 'restoration time'. The afore said distribution is obtained as a marginal distribution of the joint distribution of the number of customers in the system at time t and the number of times system reaches its capacity in time t under the conditions of catastrophes and restorations. Some measures of effectiveness are obtained explicitly with catastrophic and restoration effects.

Keywords: Catastrophes, Heterogeneous servers, Markovian queue, Restoration, Finite capacity.

1. INTRODUCTION

In the last one decade a lot of work has been done by various authors taking into consideration the concept of catastrophe B.K. Kumar *et. al.* [1] obtained the transient solution of an M/M/2 queue with heterogeneous servers subject to catastrophes. DI Crescenzo *et. al.* [3] made a continuous approximation of M/M/1 queue with catastrophe. In all the above mentioned studies the researchers have obtained the state probabilities in one way or the other and have computed various measures of performance. In this paper, the occurrence of a catastrophe makes the system empty instantly whenever the system is not empty but the system takes its own time to ready to accept new customers, this time is referred to as 'the restoration time'. The system subjected to catastrophes must take some time for its restoration after the occurrence of a catastrophe. We have obtained explicitly the distribution of the number of times the system reaches its capacity in time t under the effects of catastrophe and restoration. Various other measures of performance have also been obtained explicitly.

2. MODEL DESCRIPTION

We consider an M/M/2/N queueing system having two heterogeneous servers with FCFS discipline subject to catastrophes and restorations. The customers arrive at a counter in accordance with a Poisson process with mean arrival rate $\lambda>0$. This model is based on the following assumptions:-

1. There are two servers: server 1, the fast server and server 2, the slow server. The service times are exponentially distributed with mean service rates μ_1 and μ_2 respectively and $\mu_1>\mu_2$.
2. If the system is empty, a customer joins the fastest server with probability p and the slow with probability 1-p.
3. Each customer is served only at one server.
4. When the system is not empty, the catastrophes occur according to Poisson process with mean rate ξ.The occurrence of a catastrophe destroys all the customers in the instants and affects the system as well. The system will require some sort of time to restarts in a normal way, which is taken as restoration time. The restoration times are independently, identically exponentially distributed with parameter $\beta>0$. The customers arrive in the system during the restoration time as usual.

We define

$P_{m,n}(t)$ = Prob. [X(t) = m, Y(t) = n], $2 \le n \le N$ (1)

Where

X (t) = the number of times the system reaches its capacity in time t.

Y (t) = the number of customers in the system at time t.

$P_{m,0}(t)$ = the prob. that the system is empty at time t.

$P_{m,00}(t)$ = The prob. that there are zero customers in the system at time t without the occurrence of
Catastrophe.

$Q_{m,00}(t)$ = The prob. that there are zero customers in the system at time t with the occurrence of
catastrophe destroying all the customers.

$P_{m,1,0}(t)$ = the probability that there is one customer in the system and he is served by server 1.

$P_{m,0,1}(t)$ = the probability that there is one customer in the system and he is served by server 2.

The marginal probabilities are

$P_{.,n}(t) = \sum_{m=0}^{\infty} P_{m,n}(t)$ And $P_{m,.}(t) = \sum_{n=0}^{N} P_{m,n}(t)$

3. DIFFERENTIAL-DIFFERENCE EQUATIONS GOVERNING THE SYSTEM

$$\frac{d}{dt}P_{m,00}(t) = -\lambda P_{m,00}(t) + \mu_1 P_{m,1,0}(t) + \mu_2 P_{m,0,1}(t) + \beta Q_{m,00}(t) \quad ; n = 0, m \geq 0 \tag{2}$$

$$\frac{d}{dt}Q_{m,00}(t) = -(\lambda + \beta)Q_{m,00}(t) + \xi \sum_{n=1}^{N} P_{m,n}(t) \quad ; n = 0, m \geq 0 \tag{3}$$

$$\frac{d}{dt}P_{m,1,0}(t) = -(\lambda + \mu_1 + \xi)P_{m,1,0}(t) + \lambda p P_{m,0}(t) + \mu_2 P_{m,2}(t) \quad ; n = 1, m \geq 0 \tag{4}$$

$$\frac{d}{dt}P_{m,0,1}(t) = -(\lambda + \mu_2 + \xi)P_{m,0,1}(t) + \lambda(1-p)P_{m,0}(t) + \mu_1 P_{m,2}(t) \quad ; n = 1, m \geq 0 \tag{5}$$

$$\frac{d}{dt}P_{m,n}(t) = -(\lambda + \mu_1 + \mu_2 + \xi)P_{m,n}(t) + \lambda P_{m,n-1}(t) + (\mu_1 + \mu_2)P_{m,n+1}(t) \quad ; 2 \leq n \leq N-1, m \geq 0 \tag{6}$$

$$\frac{d}{dt}P_{m,N}(t) = -(\mu_1 + \mu_2 + \xi)P_{m,N}(t) + \lambda P_{m-1,N-1}(t) \ ; n = N, m \geq 1 \tag{7}$$

$$\frac{d}{dt}P_{0,N}(t) = 0$$

Taking Laplace transform of the equations (2)-(7) w.r.t. t we have

$$s\bar{P}_{0,00}(s) = -\lambda \bar{P}_{0,00}(s) + \mu_1 \bar{P}_{0,1,0}(s) + \mu_2 \bar{P}_{0,0,1}(s) + \beta \bar{Q}_{0,00}(s) + 1, m = 0 \tag{8}$$

$$s\bar{Q}_{0,00}(s) = -(\lambda + \beta)\bar{Q}_{0,00}(s) + \xi[\bar{P}_{0,.}(s) - \bar{P}_{0,00}(s)], m = 0 \tag{9}$$

$$s\bar{P}_{m,00}(s) = -\lambda \bar{P}_{m,00}(s) + \mu_1 \bar{P}_{m,1,0}(s) + \mu_2 \bar{P}_{m,0,1}(s) + \beta \bar{Q}_{m,00}(s), m \geq 1 \tag{10}$$

$$s\bar{Q}_{m,00}(s) = -(\lambda + \beta)\bar{Q}_{m,00}(s) + \xi[\bar{P}_{m,.}(s) - \bar{P}_{m,00}(s)], m \geq 1 \tag{11}$$

$$s\bar{P}_{m,1,0}(s) = -(\lambda + \mu_1 + \xi)\bar{P}_{m,1,0}(s) + \lambda p \bar{P}_{m,0}(s) + \mu_2 \bar{P}_{m,2}(s), n = 1 \tag{12}$$

$$s\bar{P}_{m,0,1}(s) = -(\lambda + \mu_2 + \xi)\bar{P}_{m,0,1}(s) + \lambda(1-p)\bar{P}_{m,0}(s) + \mu_1 \bar{P}_{m,2}(s) \tag{13}$$

$$(s + \lambda + \mu_1 + \mu_2 + \xi)\bar{P}_{m,n}(s) = \lambda \bar{P}_{m,n-1}(s) + (\mu_1 + \mu_2)\bar{P}_{m,n+1}(s) \ ; n = 2,3,\ldots,N-1 \tag{14}$$

$$(s + \mu_1 + \mu_2 + \xi)\bar{P}_{m,N}(s) = \lambda \bar{P}_{m-1,N-1}(s) \ ; n = N, m \geq 1 \tag{15}$$

$$s\bar{P}_{0,N}(s) = 0$$

Since $P_{0,00}(0) = 1$

Where

$$\bar{P}_{m,n}(s) = \int_0^\infty e^{-st} P_{m,n}(t)\,dt$$

Define the probability generating functions by

$$\bar{P}_n(x,s) = \sum_{m=0}^{\infty} \bar{P}_{m,n}(s) x^m \tag{16}$$

$$\bar{H}(x,y;s) = \bar{R}_0(x,s) + \sum_{n=2}^{N} \bar{P}_n(x,s)y^n$$
$$\bar{R}_0(x,s) = \bar{P}_0(x,s) + \bar{P}_{1,0}(x,s) + \bar{P}_{0,1}(x,s) \quad (17)$$

$$\bar{P}_{.}(x,s) = \sum_{m=0}^{\infty} \bar{P}_{m,.}(s)x^m \quad (18)$$

Multiplying equation (8) to (15) by x^m, summing over the ranges of m and using (16), we have

$$(s+\lambda+\xi)\bar{P}_0(x,s) = \mu_1 \bar{P}_{1,0}(x,s) + \mu_2 \bar{P}_{0,1}(x,s) + \xi \bar{P}_{.}(x,s) + 1, n = 0 \quad (19)$$

$$(s+\lambda+\mu_1+\xi)\bar{P}_{1,0}(x,s) = \lambda p \bar{P}_0(x,s) + \mu_2 \bar{P}_2(x,s) \quad (20)$$

$$(s+\lambda+\mu_2+\xi)\bar{P}_{0,1}(x,s) = \lambda(1-p)\bar{P}_0(x,s) + \mu_1 \bar{P}_2(x,s) \quad (21)$$

$$(s+\lambda+\mu_1+\mu_2+\xi)\bar{P}_n(x,s) = \lambda \bar{P}_{n-1}(x,s) + (\mu_1+\mu_2)\bar{P}_{n+1}(x,s), 2 \le n \le N-1 \quad (22)$$

$$(s+\mu_1+\mu_2+\xi)\bar{P}_N(x,s) = \lambda x \bar{P}_{N-1}(x,s) \quad (23)$$

Using (19) to (21), we get

$$\bar{P}_{1,0}(x,s) = [\mu_2\{2(s+\lambda+\xi)+\mu_1+\mu_2\}]^{-1} \begin{bmatrix} [(s+\lambda+\xi)(s+\mu_1+\lambda+\xi) - \lambda\{p(\mu_1+\mu_2)-\mu_2\}]\bar{P}_0(x,s) \\ -s^{-1}(s+\mu_1+\lambda+\xi)\{(s+\xi)+\lambda\xi(x-1)\bar{P}_{N-1}(x,s)\} \end{bmatrix} \quad (24)$$

$$\bar{P}_{0,1}(x,s) = [\mu_1\{2(s+\lambda+\xi)+\mu_1+\mu_2\}]^{-1} \begin{bmatrix} [(s+\lambda+\xi)(s+\mu_2+\lambda+\xi) + \lambda\{p(\mu_1+\mu_2)-\mu_2\}]\bar{P}_0(x,s) \\ -s^{-1}(s+\mu_2+\lambda+\xi)\{(s+\xi)+\lambda\xi(x-1)\bar{P}_{N-1}(x,s)\} \end{bmatrix} \quad (25)$$

Multiplying equation (22) to (23) by y^n, summing over the ranges of n and using (17), we have on simplification

$$\bar{H}(x,y;s) = \frac{D_1}{D_2} \quad (26)$$

Where

$$D_1 = y[1+\xi s^{-1}] + (1-y)(\lambda y - \mu_1 - \mu_2)\bar{R}_0(x,s) - \lambda y(1-y^2)[\bar{P}_{1,0}(x,s) + \bar{P}_{0,1}(x,s)] + y(1-y)\bar{P}_2(x,s) + \lambda\left[s^{-1}y\xi(x-1) + y^{N+1}\left\{(x-1) + \frac{\lambda x(1-y)}{s+\mu_1+\mu_2+\xi}\right\}\right]\bar{P}_{N-1}(x,s)$$

$$D_2 = [(s+\xi)y + (1-y)(\lambda y - (\mu_1+\mu_2))]$$

$$\bar{P}_N(x,s) = \frac{\lambda x(1-y)}{s+\mu_1+\mu_2+\xi}\bar{P}_{N-1}(x,s)$$

$$\alpha_i(s) = \frac{(s+\lambda+\mu_1+\mu_2+\xi) \pm \sqrt{(s+\lambda+\mu_1+\mu_2+\xi)^2 - 4\times\lambda(\mu_1+\mu_2)}}{2\lambda}, i = 1, 2.$$

The existence of $\bar{H}(x, y; s)$ is only possible if numerator vanishes for α_1 and α_2 the two zeros of the denominators. This will give rise two equations, solving them we have;

$$\bar{P}_0(x,s) = \frac{\bar{E}_1(x,s)}{\bar{E}(x,s)} \qquad (27)$$

$$\bar{P}_{N-1}(x,s) = \frac{\bar{E}_2(.,s)}{\bar{E}(x,s)} \qquad (28)$$

Where

$$\bar{E}_1(x,s) = \begin{bmatrix} \lambda(s+\xi)A_1 x\gamma^{-2}[s(s+\mu_1+\mu_2+\xi)]^{-1}\left[\{\lambda(A_1-1)-(s+\mu_1+\xi)\}T(N)+(s+\mu_1+\lambda+\xi)T(N-1)\right] \\ +\lambda^2 A_1 A_2 x[s(s+\mu_1+\mu_2+\xi)]^{-1}[\lambda\gamma^{-4}\{T(N-1)-T(N-2)\}+(\mu_1+\mu_2)\{T(N+1)-T(N)\}]+ \\ s^{-2}[\lambda\xi(x-1)(s+\xi)A_2[\mu_1+\mu_2-\lambda\gamma^{-2}][A_1-\gamma^{-2}(s+\mu_1+\lambda+\xi)]]+\lambda\gamma^{-2}A_1(x-1)[\mu_1+\mu_2+s^{-2} \\ (s+\xi)(s+\mu_1+\lambda+\xi)]T(N-1)+\lambda\xi(x-1)A_2 s^{-2}[\lambda\gamma^{-2}V(2)\{A_1(s+\xi)+(s+\mu_1+\lambda+\xi)\}]+s^{-2} \\ \lambda\gamma^{-2}A_1A_2V(2)(\mu_1+\mu_2) \end{bmatrix}$$

$$\bar{E}_2(.,s) = s^{-1}\begin{bmatrix} [(s+\xi)\{A_1-(s+\mu_1+\lambda+\xi)\}-A_2(\mu_1+\mu_2)][V(1)\{\gamma^{-2}(B_1+\lambda B_2)-B_2(\mu_1+\mu_2)\}-\lambda\gamma^{-2}B_2V(2)] \\ +[(s+\xi)(s+\mu_1+\lambda+\xi)+A_2\lambda][\gamma^{-2}V(1)\{B_1-\lambda\gamma^{-2}B_2+B_2(\mu_1+\mu_2)\}-V(2)B_2(\mu_1+\mu_2)]+A_2 \\ (\mu_1+\mu_2)[-V(1)\{B_1+B_2(\mu_1+\mu_2)\}-V(2)\{B_1-\lambda B_2\}-V(3)\lambda B_2] \end{bmatrix}$$

$$\bar{E}(x,s) = \begin{bmatrix} \lambda\xi(x-1)s^{-1}[\{\lambda A_2+(s+\mu_1+\lambda+\xi)\}\{\lambda\gamma^{-4}B_2V(1)+B_2(\mu_1+\mu_2)V(2)-\gamma^{-2}V(1)\{B_1+B_2(\mu_1+\mu_2)\}\} \\ +\{A_1-A_2(\mu_1+\mu_2)-(s+\mu_1+\lambda+\xi)\}\{B_2(\mu_1+\mu_2)V(1)+\lambda\gamma^{-2}B_2V(2)-\gamma^{-2}\{B_1+\lambda B_2\}\}]+[\lambda(x-1) \\ A_1+\lambda^2A_1x(s+\mu_1+\mu_2+\xi)^{-1}][(\mu_1+\mu_2)B_2V(N+1)-\lambda\gamma^{-6}B_2V(N-2)-\gamma^{-2}V(N)\{B_1+B_2(\mu_1+\mu_2)\} \\ +\gamma^{-4}V(N-1)\{B_1+\lambda B_2\}]-\lambda^2A_1x(s+\mu_1+\mu_2+\xi)^{-1}[(\mu_1+\mu_2)B_2V(N+2)-\lambda\gamma^{-6}B_2V(N-1)-\gamma^{-2} \\ V(N+1)\{B_1+B_2(\mu_1+\mu_2)\}+\gamma^{-4}V(N)\{B_1+\lambda B_2\}] \end{bmatrix}$$

$$A_1 = \mu_1\mu_2\left[(s+\mu_1+\lambda+\xi)+(s+\mu_2+\lambda+\xi)\right]$$

$$A_2 = [(\mu_1+\mu_2)(s+\lambda+\xi)+\mu_1{}^2+\mu_2{}^2]$$

$$B_1 = [(s+\mu_1+\lambda+\xi)\{(s+\lambda+\xi)^2+(s+\lambda p+\xi)\mu_2\}-\lambda p\mu_1(s+\mu_2+\lambda+\xi)]$$

$$B_2 = [(s+\lambda+\xi)A_2+(\mu_1+\mu_2)\{\mu_1\mu_2+\lambda p(\mu_2-\mu_1)\}-\lambda\mu_2(\mu_2-\mu_1)]$$

$$\alpha_1\alpha_2 = \frac{\mu_1+\mu_2}{\lambda} = \gamma^{-2}(say), \quad T(n) = V(n+1)-\gamma^{-2}V(n)$$

and

$V(r) = \alpha_1^r(s) - \alpha_2^r(s), r = 0, 1, 2, 3, ...$

If we write

$D_2 = -\lambda[(\alpha_1 - y)(\alpha_2 - y)]$

(26) yields

$$\frac{D_1}{\lambda V(1)}\left[\alpha_1^{-1}\sum_{n=0}^{\infty}\left(\frac{y}{\alpha_1}\right)^n - \alpha_2^{-1}\sum_{n=0}^{\infty}\left(\frac{y}{\alpha_2}\right)^n\right], \quad \left|\frac{y}{\alpha_2}\right| < 1 \tag{29}$$

Now $\bar{P}_n(x,s)$ is the coefficient of y^n in (17). Comparing the coefficients of y^n on both sides of (29), we have:

$$\bar{P}_n(x,s) = \frac{\gamma^{2n}}{\lambda V(1)}\left[s^{-1}F_1 + F_2\,\bar{P}_0(x,s) + F_3\,\bar{P}_{N-1}(x,s)\right] \tag{30}$$

Where

$$F_1 = [(s+\xi)\{T(n-1)(s+\lambda+\mu_1+\xi) - A_1V(n)\} - A_2\{\gamma^2T(n)(\mu_1+\mu_2) + \lambda\gamma^{-2}T(n-2)\}]$$

$$F_2 = \begin{bmatrix} T(n-1)[(s+\lambda+\mu_1+\xi)\{(s+\lambda+\xi)^2 + (s+\lambda p+\xi)\mu_2\} - \lambda p\mu_1(s+\lambda+\mu_2+\xi)] \\ +[\gamma^2T(n)(\mu_1+\mu_2) + \lambda\gamma^{-2}T(n-2)][(s+\lambda+\xi)A_2 + (\mu_1+\mu_2)\{\lambda p(\mu_2-\mu_1) + \mu_1\mu_2\} \\ -\lambda\mu_2(\mu_2-\mu_1)] \end{bmatrix}$$

$$F_3 = \left[s^{-1}\lambda\xi(x-1)[T(n-1)(s+\lambda+\mu_1+\xi) - \{\gamma^2T(n)(\mu_1+\mu_2) + \lambda\gamma^{-2}T(n-2) + V(n)\}A_2]\right]$$

Since $\bar{P}_0(x,s)$ and $\bar{P}_{N-1}(x,s)$ are given by (27) and (28), so $\bar{P}_n(x,s)$ are explicitly known.

For setting x=0 in (30) gives

$$\bar{P}_{0,n}(s) = \frac{\gamma^{2n}}{\lambda s^2 Z_6 V(1)}\left[sZ_1Z_6 + Z_2\{Z_3 - Z_7\} - \lambda\xi Z_4Z_5\right] \tag{31}$$

Where

$$Z_1 = \left[(s+\xi)\{T(n-1)(s+\mu_1+\lambda+\xi) - A_1V(n)\} - A_2\{\gamma^2T(n)(\mu_1+\mu_2) + \lambda\gamma^{-2}T(n-2)\}\right]$$

$$Z_2 = \begin{bmatrix} T(n-1)[(s+\mu_1+\lambda+\xi)\{(s+\lambda+\xi)^2 + (s+\lambda p+\xi)\mu_2\} - \lambda p\mu_1(s+\mu_2+\lambda+\xi)] + [\gamma^2T(n)(\mu_1+\mu_2) + \\ \lambda\gamma^{-2}T(n-2)][(s+\lambda+\xi)A_2 + (\mu_1+\mu_2)\{\mu_1\mu_2 + \lambda p(\mu_2-\mu_1)\} - \lambda\mu_2(\mu_2-\mu_1)] \end{bmatrix}$$

$$Z_3 = \left[-\lambda\xi(s+\xi)[\mu_1+\mu_2-\lambda\gamma^{-2}][A_1 - \gamma^{-2}(s+\mu_1+\lambda+\xi)]\right]$$

$$Z_4 = \left[T(n-1)(s+\mu_1+\lambda+\xi) - \{\gamma^2T(n)(\mu_1+\mu_2) + \lambda\gamma^{-2}T(n-2) + V(n)\}A_2\right]$$

$$Z_5 = \begin{bmatrix} [(s+\xi)\{A_1 - (s+\mu_1+\lambda+\xi)\} - A_2(\mu_1+\mu_2)][V(1)\{\gamma^{-2}(B_1+\lambda B_2) - B_2(\mu_1+\mu_2)\} - \lambda B_2\gamma^{-2}V(2)] + \\ [\lambda A_2 + (s+\xi)(s+\mu_1+\lambda+\xi)][V(1)\gamma^{-2}\{(B_1 - \lambda\gamma^{-2}B_2 + B_2(\mu_1+\mu_2)\} - B_2(\mu_1+\mu_2)V(2)] + A_2(\mu_1+\mu_2) \\ [-V(1)\{B_1 + B_2(\mu_1+\mu_2)\} - V(2)\{B_1+\lambda B_2\} - \lambda B_2V(3)] \end{bmatrix}$$

$$Z_6 = [-\lambda\xi s^{-1}]\begin{bmatrix} [\lambda A_2 + (s+\mu_1+\lambda+\xi)][\lambda\gamma^{-4}B_2V(1) + (\mu_1+\mu_2)B_2V(2) - \gamma^{-2}V(1)\{B_1 + B_2(\mu_1+\mu_2)\}] + [A_1 - A_2 \\ (\mu_1+\mu_2) - (s+\mu_1+\lambda+\xi)][(\mu_1+\mu_2)B_2V(1) + \lambda\gamma^{-2}B_2V(2) - \gamma^{-2}\{B_1+\lambda B_2\} + \gamma^{-4}V(N-1) \\ \{B_1+\lambda B_2\} + [-\lambda A_1\{(\mu_1+\mu_2)B_2V(N+1) - \lambda\gamma^{-6}B_2V(N-2)\} - \gamma^{-2}V(N)\{B_1 + B_2(\mu_1+\mu_2)\}] \end{bmatrix}$$

$$Z_7 = \left[-\lambda\gamma^{-2}A_1[(s+\xi)(s+\mu_1+\lambda+\xi) + s^2(\mu_1+\mu_2)]T(N-1) + \lambda\gamma^{-2}A_1A_2(\mu_1+\mu_2)V(2)\right]$$

Applying the Leibniz differentiation theorem to (24) to (25) and (27), setting x=0 and dividing both sides by m! We have

$$\bar{P}_{m,1,0}(s) = \frac{Z_8}{Z_9}, m \geq 1 \tag{32}$$

$$\bar{P}_{m,0,1}(s)=\frac{Z_{10}}{Z_{11}}, m\geq 1 \tag{33}$$

$$\bar{P}_{m,0}(s)=\frac{Z_{12}}{Z_{13}}, m\geq 1 \tag{34}$$

Where

$$Z_8=\begin{bmatrix}[s\lambda\{(\mu_1+\mu_2)p-\mu_2\}A_1A_2V(2)-[\lambda\xi(s+\lambda+\mu_2+\xi)+(s+\xi)(\mu_1+\mu_2)(s+\lambda+\mu_2+\xi)]T(N-1)+s(s+\xi)\\ (s+\lambda+\mu_2+\xi)A_1A_2{}^2+[2\lambda\mu_2A_1+2\lambda^2A_1A_2\{(\mu_1+\mu_2)p-\mu_2\}B_2V(N+1)-\lambda\gamma^{-6}V(N-2)B_2-\gamma^{-2}V(N)\\ \{B_1+(\mu_1+\mu_2)B_2\}]][\lambda^2A_1(\mu_1+\mu_2)B_2\{V(N+2)-V(N+1)\}+A_1A_2B_1\gamma^2\{V(N+1)-V(N)\}+\gamma^{-4}\{B_1+\lambda B_2\}\\ \{T(N-1)-T(N-2)\}+\lambda^{-1}\gamma^{-4}(s+\xi)(s+\lambda+\mu_2+\xi)\{\lambda T(N)(A_1-1)-T(N-1)\}][2\lambda A_1A_2\gamma^{-4}\{\{(\mu_1+\mu_2)p-\mu_2\}B_1\\ +B_2\}\{T(N-1)-T(N-2)\}+\gamma^{-4}\{B_1+\lambda B_2\}V(N)+(\mu_1+\mu_2)p\{T(N-1)-T(N)\}-2\lambda^2A_1\{(\mu_1+\mu_2)B_2V(N+2)-\lambda\gamma^{-6}\\ B_2V(N-1)-\gamma^{-2}V(N+1)\}B_1A_2+\lambda\xi A_2V(2)\{A_1(s+\xi)+\lambda^2\{(\mu_1+\mu_2)p-\mu_2\}\}]^{(m-1)}\end{bmatrix}$$

$$Z_9=\begin{bmatrix}[\gamma^{2m+2}s^2(s+\mu_1+\mu_2+\xi)^m][s^{-(m+1)}(\lambda\xi)^{m+1}\mu_1\{2(s+\lambda+\xi)+\mu_1+\mu_2\}][2\lambda A_1[(\mu_1+\mu_2)B_2V(N+1)-\lambda\gamma^{-6}V(N-2)B_2\\ -\gamma^{-2}V(N)\{B_1+(\mu_1+\mu_2)B_2\}-\gamma^{-4}V(N-1)\{B_1+\lambda B_2\}-\{A_1-A_2(\mu_1+\mu_2)-(s+\lambda+\mu_1+\xi)\}\{(\mu_1+\mu_2)B_2V(1)+\lambda B_2\\ \gamma^{-2}V(2)-\gamma^{-2}\{B_1+\lambda B_2\}\}]-[\lambda A_2+(s+\lambda+\mu_1+\xi)][\gamma^{-4}\lambda B_2V(1)+B_2V(2)(\mu_1+\mu_2)-\gamma^{-2}V(1)\{B_1+(\mu_1+\mu_2)B_2\}]]^{m+1}\end{bmatrix}$$

$$Z_{10}=\begin{bmatrix}[s\lambda(\mu_1+\mu_2)pA_1A_2V(2)-\{\lambda\xi(s+\lambda+\mu_1+\xi)+(s+\xi)(\mu_1+\mu_2)(s+\lambda+\mu_1+\xi)\}]T(N-1)+s(s+\xi)\\ (s+\lambda+\mu_1+\xi)A_1A_2{}^2+[2\lambda A_1\mu_2+2\lambda^2A_1A_2\{(\mu_1+\mu_2)pB_2V(N+1)-\lambda\gamma^{-6}V(N-2)B_2-\gamma^{-2}V(N)\{B_1+\\ (\mu_1+\mu_2)B_2\}\}][\lambda^2A_1(\mu_1+\mu_2)B_2\{V(N+2)-V(N+1)\}+A_1A_2B_1\gamma^{-2}\{V(N+1)-V(N)\}+\gamma^{-4}\{B_1+\lambda B_2\}\\ \{T(N-1)-T(N-2)\}+\lambda^{-1}\gamma^{-4}(s+\xi)(s+\lambda+\mu_2+\xi)\{\lambda T(N)(A_1-1)-T(N-1)\}][2\lambda A_1A_2\gamma^{-4}\{(\mu_1+\mu_2)pB_1+B_2\}\\ \{T(N-1)-T(N-2)\}+\gamma^{-4}\{B_1+\lambda B_2\}V(N)+(\mu_1+\mu_2)p\{T(N-1)-T(N)\}-2\lambda^2A_1\{(\mu_1+\mu_2)B_2V(N+2)\\ -\lambda\gamma^{-6}B_2V(N-1)-\gamma^{-2}V(N+1)\}B_1A_2+\lambda\xi A_2V(2)\{A_1(s+\xi)+\lambda^2\{(\mu_1+\mu_2)p-\mu_2\}\}]^{(m-1)}\end{bmatrix}$$

$$Z_{11}=\begin{bmatrix}[\gamma^{2m+2}s^2(s+\mu_1+\mu_2+\xi)^m][s^{-(m+1)}(\lambda\xi)^{m+1}\mu_2\{2(s+\lambda+\xi)+\mu_1+\mu_2\}][2\lambda A_1[(\mu_1+\mu_2)B_2V(N+1)-\lambda\gamma^{-6}V(N-2)B_2\\ -\gamma^{-2}V(N)\{B_1+(\mu_1+\mu_2)B_2\}-\gamma^{-4}V(N-1)\{B_1+\lambda B_2\}-\{A_1-A_2(\mu_1+\mu_2)-(s+\lambda+\mu_1+\xi)\}\{(\mu_1+\mu_2)B_2V(1)+\lambda B_2\\ \gamma^{-2}V(2)-\gamma^{-2}\{B_1+\lambda B_2\}\}]-[\lambda A_2+(s+\lambda+\mu_1+\xi)][\gamma^{-4}\lambda B_2V(1)+B_2V(2)(\mu_1+\mu_2)-\gamma^{-2}V(1)\{B_1+(\mu_1+\mu_2)B_2\}]]^{m+1}\end{bmatrix}$$

$$Z_{12}=\begin{bmatrix}A_1A_2V(2)(\mu_1+\mu_2)-A_1[(s+\xi)(s+\lambda+\mu_2+\xi)+(\mu_1+\mu_2)]T(N-1)+[2A_1\lambda+2\lambda^2A_2\{(\mu_1+\mu_2)B_2V(N+1)\\ -\lambda\gamma^{-6}B_2V(N-2)-\gamma^{-2}V(N)\{B_1+(\mu_1+\mu_2)B_2\}\}][2\lambda\gamma^2V(1)+(\mu_1+\mu_2)T(N)+\gamma^2\{2(\mu_1+\mu_2)(s+\lambda+\xi)\}\\ T(N-1)-\lambda^2\gamma^2V(N+2)][2\lambda A_1A_2\gamma^{-4}\{B_1+(\mu_1+\mu_2)B_2\}\{T(N-1)-T(N-2)\}+(\mu_1+\mu_2)\{T(N-1)-T(N)\}\\ +\gamma^{-4}V(N)\{B_1+\lambda B_2\}-2\lambda^2A_1\{(\mu_1+\mu_2)B_2V(N+2)-\lambda\gamma^{-6}B_2V(N-1)-\gamma^{-2}V(N+1)\}B_1A_2+\lambda\xi A_2V(2)\{A_1(s+\xi)\\ +\lambda^2(\mu_1+\mu_2)\}]^{(m-1)}\end{bmatrix}$$

$$Z_{13}=\begin{bmatrix}[\gamma^{2m+2}s(s+\mu_1+\mu_2+\xi)^m][s^{-(m+1)}(\lambda\xi)^{m+1}][2\lambda A_1[(\mu_1+\mu_2)B_2V(N+1)-\lambda\gamma^{-6}V(N-2)B_2-\gamma^{-2}V(N)\{B_1+(\mu_1+\mu_2)B_2\}\\ -\gamma^{-4}V(N-1)\{B_1+\lambda B_2\}-\{A_1-A_2(\mu_1+\mu_2)-(s+\lambda+\mu_1+\xi)\}\{(\mu_1+\mu_2)B_2V(1)+\lambda B_2\gamma^{-2}V(2)-\gamma^{-2}\{B_1+\lambda B_2\}\}]-\\ [\lambda A_2+(s+\lambda+\mu_1+\xi)][\gamma^{-4}\lambda B_2V(1)+B_2V(2)(\mu_1+\mu_2)-\gamma^{-2}V(1)\{B_1+(\mu_1+\mu_2)B_2\}]]^{m+1}\end{bmatrix}$$

Now $$\bar{P}_{m,n}(s)=\frac{\gamma^{2n}}{\lambda V(1)}\left[Z_{14}\,\bar{P}_{m,0}(s)\right] \tag{35}$$

Where

$$Z_{14}=\begin{bmatrix}s^{-1}[(s+\xi)\{T(n-1)(s+\lambda+\mu_1+\xi)-A_1V(n)\}-A_2\{\gamma^2T(n)(\mu_1+\mu_2)+\lambda\gamma^{-2}T(n-2)\}]+[T(n-1)\{(s+\lambda+\mu_1+\xi)\\ \{(s+\lambda+\xi)^2+(s+\lambda p+\xi)\mu_2\}-\lambda p\mu_1(s+\lambda+\mu_2+\xi)\}+[\gamma^2T(n)(\mu_1+\mu_2)+\lambda\gamma^{-2}T(n-2)][(s+\lambda+\xi)+(\mu_1+\mu_2)\\ \{\lambda p(\mu_2-\mu_1)+\mu_1\mu_2\}-\lambda\mu_2(\mu_2-\mu_1)]]\end{bmatrix}$$

Using (10) and (11), we have

$$\bar{P}_{m,00}(s)=\frac{(s+\lambda+\beta)\bar{P}_{m,0}(s)}{(s+\lambda+\beta-\xi)}-\frac{\xi[\sum_{n=0}^{N}\bar{P}_{m,n}(s)]}{(s+\lambda+\beta-\xi)} \tag{36}$$

$$\bar{Q}_{m,00}(s) = \frac{\xi[\sum_{n=0}^{N} \bar{P}_{m,n}(s)]}{(s+\lambda+\beta-\xi)} - \frac{\xi \bar{P}_{m,0}(s)}{(s+\lambda+\beta-\xi)} \tag{37}$$

We know that

$$(1-a/b)^{-(j+1)} = \sum_{i=0}^{\infty}\left(C_i^{\,j+i}\right)\frac{a^i}{b^i}, (a+b)^{j+1} = a(a+b)^j + b(a+b)^j, (a-b)^j = \sum_{i=0}^{j}(-1)^i\left(C_i^{\,j}\right)a^{j-i}b^i$$

and $(1\text{-a})^{\text{-j}} = \sum_{k=0}^{\infty}\left(C_k^{\,j+k-1}\right)a^k$

Using these Identities in (31) to (37), we have

$$\bar{P}_{0,n}(s) = \left[\begin{aligned}
&\sum_{j=0}^{\infty}\sum_{l=0}^{j+1}\sum_{k=0}^{\infty}\sum_{h=0}^{l}\sum_{r=0}^{\infty}\sum_{p1=0}^{h}\sum_{q=0}^{j-l}(-1)^{l+h+p1+q}\binom{j+1}{l}\binom{l}{h}\binom{j+l+r}{r}\binom{h}{p1}\binom{j-l}{q}\lambda^{-(j+l-h+p1+q+2)}\gamma^{-2(j+k+h-l+1)} \\
&\Big[\{2(\mu_1+\mu_2)+\lambda p(\mu_2-\mu_1)+\mu_1\mu_2\}[\gamma^{-1}\{R^{-(g+j+h-l-n-1)}-R^{-(g+j+h-l+n+1)}\}-\gamma^{-2}\{R^{-(g+j+h-l-n+1)} \\
&-R^{-(g+j+h-l+n+1)}\}]+(\mu_1+\mu_2)^{l+h}\gamma^{-2(k+h+l+1)}(s+\lambda p+\xi)^{-(j+k-l+h-p1)}[R^{-(g+j+h+l+k+1)}-\gamma\{R^{-(g+j+h+l+k)} \\
&-R^{-(g+j+h+l+k+2)}\}]\Big]+[\lambda^2\xi\{p(\mu_2-\mu_1)+(\lambda+\xi+\mu_1)(\lambda+\xi+\mu_2)(\mu_1+\mu_2)-\lambda\mu_2^{\,2}(1-p)+\mu_1\mu_2\lambda \\
&-\lambda\mu_1^{\,2}p\}]\left[\begin{aligned}&\sum_{j=0}^{\infty}\sum_{l=0}^{j+1}\sum_{k=0}^{\infty}\sum_{h=0}^{l}\sum_{r=0}^{\infty}(-1)^{l+h}\binom{j+l}{l}\binom{l}{h}\binom{j+l+r}{r}2^{-(j+1)}\lambda^{-(j+k+h-l+2)}\gamma^{-2(j+k+h-l)}[\{R^{-(g+j+h-l-n-1)} \\
&-R^{-(g+j+h-l+n+1)}\}-\gamma^{-1}\{R^{-(g+j+h-l-n)}-R^{-(g+j+h-l+n)}\}]+(s+\lambda+\xi)^{-(k-l)}[\{R^{-(g+j+h-l-n)}- \\
&R^{-(g+j+h-l+n)}\}-\gamma^{-1}\{R^{-(g+j+h-l-n+1)}-R^{-(g+j+h-l+n-1)}\}]\end{aligned}\right] \\
&+\xi(\mu_1+\mu_2)[(\lambda+\xi+\mu_1)(\lambda+\xi+\mu_2)+\mu_1\mu_2\{2(\lambda+\xi)+(\mu_1+\mu_2)\}](s+\lambda+\mu_1+\xi)^{-3}\Bigg[\{\sum_{i=0}^{N-2}\gamma^{-2(i+1)} \\
&(2\lambda)^{2i-N+2}[(s+\lambda+\mu_1+\mu_2+\xi)+\nu]^{-(2i-N+2)}\}-\sum_{i=0}^{N-1}\gamma^{-2i}(2\lambda)^{2i-N+1}[(s+\lambda+\mu_1+\mu_2+\xi)+\nu]^{-(2i-N+1)}\Bigg] \\
&+\gamma^{-2}\lambda^{-1}[-\lambda\{(\lambda+\xi)(\lambda+\xi+\mu_1)(\lambda+\xi+\mu_2)\}-\lambda^2\mu_1 p(\lambda+\xi+\mu_2)-\mu_2(1-p)\lambda(\lambda+\xi+\mu_1)] \\
&\Bigg[\{\sum_{i=0}^{n-2}\gamma^{2(n-i-1)}(2\lambda)^{2i-n+2}[(s+\lambda+\mu_1+\mu_2+\xi)+\nu]^{-(2i-n+2)}\}-3\{\sum_{i=0}^{n-1}\gamma^{2(n-i)}(2\lambda)^{2i-n+1} \\
&[(s+\lambda+\mu_1+\mu_2+\xi)+\nu]^{-(2i-n+1)}\}+\{\sum_{i=0}^{N-2}\gamma^{-2(i+1)}(2\lambda)^{2i-N+2}[(s+\lambda+\mu_1+\mu_2+\xi)+\nu]^{-(2i-N+2)}\}\Bigg]+ \\
&\sum_{j=0}^{\infty}\sum_{l=0}^{j+1}\sum_{k=0}^{\infty}\sum_{i=0}^{N+2}(-1)^{l+i}\gamma^{-2(j+k+i+1)}\binom{j+l}{l}\binom{N+2}{i}[\{R^{-(g+j+l+i-N+2)}-R^{-(g+j+l+i-N-2)}\}]+\sum_{j=0}^{\infty}\sum_{l=0}^{j+2}\sum_{k=0}^{\infty}(-1)^l \\
&\binom{j+2}{l}\binom{j+k+1}{k}\gamma^{-(2j-l-1)}[\{R^{-(g+j+i)}-R^{-(g+j+3)}\}]-\sum_{i=0}^{N-1}\gamma^{-2i}(2\lambda)^{2i-N+1}[(s+\lambda+\mu_1+\mu_2+\xi)+\nu]^{-(2i-N+1)}
\end{aligned}\right] \tag{38}$$

$$
\bar{P}_{m,1,0}(s)=\left[\begin{array}{l}
\sum_{j=0}^{\infty}\sum_{l=0}^{j+1}\sum_{k=0}^{\infty}\sum_{h=0}^{l}\sum_{r=0}^{\infty}\sum_{p1=0}^{h}\sum_{i=0}^{m-1}(-1)^{j+l+l+h+p1}\binom{j+m}{j}\binom{j+1}{l}\binom{l}{h}\binom{j+l+r+1}{r}\binom{h}{p1}\binom{m-1}{i} \\
[\lambda p(\mu_1+\mu_2)+\mu_1\mu_2\xi^2]^{j+k+l}\gamma^{-2(m+j+k-l+1)}2^{-(j+k+p1+2)}(s+\mu_1+\mu_2+\xi)^{-(m+i+l+k+p1)}[\{R^{(g+j+l+h+i+p1+1)}- \\
R^{(g+j+l+h+i+p1+2)}\}+\gamma^{-2}\{R^{(g+j+l+h+i-l-n+1)}-R^{(g+j+l+h+i-l-n-1)}\}-(2\lambda)^{j+i+k+2}\gamma^{-6}[\lambda(\mu_1+\mu_2)+\lambda(1-p)\mu_1\mu_2 \\
+\mu_2\{(\lambda+\xi)(\lambda+\xi+\mu_2)-\lambda p\mu_1\}][R^{(g+j+l+h+i+p1-l-n)}-R^{(g+j+l+h+i+p1-l+n)}]]+[\lambda\xi p(\lambda+\xi+\mu_2)+ \\
\lambda^2\{(\mu_1+\mu_2)p-\mu_2\}+\mu_1\mu_2(1-p)\lambda+\mu_2\{(\lambda+\xi)(\lambda+\xi+\mu_2)-\lambda p\mu_1\mu_2\}][\sum_{j=0}^{\infty}\sum_{l=0}^{j+i+1}\sum_{k=0}^{\infty}\sum_{i=0}^{m-1}(-1)^{i+l} \\
\binom{j+i+l}{l}\binom{m-1}{i}\binom{j+k}{k}\lambda^{m+i+1}\mu_1^{-(l+k)}\gamma^{-(2m+j-i-1)}(s+\mu_1+\mu_2+\xi)^{-(m+i+l)}[R^{(g+j+l+n-1)}-R^{(g+j+l-n+1)}] \\
+\lambda^2\xi[(\mu_1+\mu_2)p-\mu_2](s+\lambda+\mu_2+\xi)^{-3}[R^{(g+j+l+k-n-1)}-R^{(g+j+l+k-n+1)}]]-\sum_{j=0}^{\infty}\sum_{l=0}^{j+1}\sum_{k=0}^{\infty}\sum_{i=0}^{N+2}(-1)^{l}\binom{j+l}{l} \\
[\lambda^2[(\mu_1+\mu_2)p-\mu_2]^{l+i}[(2\lambda)^{2(j+k+i)-N-2}[(s+\lambda+\mu_1+\mu_2+\xi)+\nu]^{-[2(j+k+i)-N-2]}+(s+\lambda+\mu_1+\xi)^{-(j-l+i)} \\
[R^{(g+j+k-l-n-1)}-R^{(g+j+k-l+n-1)}]]+[\lambda\xi[(\mu_1+\mu_2)p-\mu_2](s+\lambda+\mu_1+\xi)^{-3}[\sum_{i=0}^{n-1}\gamma^{-2(n-i)}(2\lambda)^{2i+n+1} \\
[(s+\lambda+\mu_1+\mu_2+\xi)+\nu]^{-(2i+n+1)}]+[2(\lambda+\xi+\mu_1)(\lambda+\xi+\mu_2)+\lambda p\mu_1\mu_2(\lambda+\xi+\mu_2)+\lambda^3 \\
[(\mu_1+\mu_2)p-\mu_2]]\sum_{j=0}^{\infty}\sum_{l=0}^{j+1}\sum_{k=0}^{\infty}\sum_{h=0}^{l}\sum_{r=0}^{\infty}(-1)^{l+h}\binom{j+1+l}{l}\binom{m+j}{j}\binom{l}{h}\gamma^{-2(j+k+r)}(s+\lambda+\xi)^{-(k+h-l)} \\
[\{R^{(g+j+h-l-n)}-R^{(g+j+h-l+n)}\}-\gamma^{-1}\{R^{(g+j+h-l-n+1)}-R^{(g+j+h-l+n+1)}\}]+\lambda\xi(s+\xi)^{-3}[\lambda(\mu_1+\mu_2)p-\mu_2]] \\
\sum_{i=0}^{N+1}\gamma^{-2(i+2)}(2\lambda)^{2i-N-1}[(s+\lambda+\mu_1+\mu_2+\xi)+\nu]^{-(2i-N-1)}+\sum_{i=0}^{N-2}\gamma^{-2(i+1)}(2\lambda)^{2i-N+2}[(s+\lambda+\mu_1+\mu_2+\xi)+\nu]^{-(2i-N+2)}]
\end{array}\right] \quad (39)
$$

$$
\bar{P}_{m,0,1}(s)=\left[\begin{array}{l}
\sum_{j=0}^{\infty}\sum_{l=0}^{j+1}\sum_{k=0}^{\infty}\sum_{h=0}^{l}\sum_{r=0}^{\infty}\sum_{p1=0}^{h}\sum_{i=0}^{m-1}(-1)^{i+l+h+p1}\binom{j+m}{j}\binom{j+1}{l}\binom{l}{h}\binom{j+l+r+1}{r}\binom{h}{p1}\binom{m-1}{i} \\
[\lambda(\mu_1+\mu_2)+\mu_1\mu_2]^{j+k+l}\gamma^{-2(m+j+k+r-l+1)}2^{-(j+k+p1+1)}(s+\mu_1+\mu_2+\xi)^{-(m-i+l+k+p1)} \\
\Big[\{R^{-(g+j+h+i+p1+1)}-\gamma^{-2}R^{-(g+j+h+i+p1+2)}\}+\{R^{-(g+j+h+i-l-n-1)}-R^{-(g+j+h+i-l-n+1)}\} \\
-(2\lambda)^{i+k+2}\gamma^{-4}[\lambda(\mu_1+\mu_2)+(1-p)\mu_1\mu_2+\mu_1\lambda\{(\lambda+\xi)(\lambda+\xi+\mu_1)-\lambda p\mu_1\}] \\
[R^{-(g+j+h+i+p1-l-n)}-R^{-(g+j+h+i+p1-l+n)}]\Big]+[\lambda\xi p(\lambda+\xi+\mu_2)+\mu_1\mu_2(1-p)\lambda+ \\
\mu_2\{(\lambda+\xi)(\lambda+\xi+\mu_2)-\lambda(1-p)\mu_2\}]\Bigg[\sum_{j=0}^{\infty}\sum_{l=0}^{j+i+1}\sum_{k=0}^{\infty}\sum_{i=0}^{m-1}(-1)^{i+l}\binom{j+i+1}{l}\binom{m-1}{i} \\
\binom{j+i+k}{k}\lambda^{m-i+1}\mu_2^{-(l+k)}\gamma^{-(2m+j-i-1)}(s+\mu_1+\mu_2+\xi)^{-(m-i+l)}[R^{-(g+j+l+n-1)}-R^{-(g+j+l-n+1)}] \\
+\xi(s+\lambda+\mu_1+\xi)^{-3}[R^{-(g+j+l+k-n-1)}-R^{-(g+j+l+k+n-1)}]\Bigg]-\sum_{j=0}^{\infty}\sum_{l=0}^{j+1}\sum_{k=0}^{\infty}\sum_{i=0}^{N-1}(-1)^{i+l}\binom{j+1}{l} \\
\binom{N-1}{i}\binom{N+j}{j}\gamma^{-(j+k+i+1)}[\lambda^2[(\mu_1+\mu_2)p]^{l+i} \\
\Big[(2\lambda)^{2(j+k+i)-N+1}[(s+\lambda+\mu_1+\mu_2+\xi)+\nu]^{-[2(j+k+i)-N+1]}+(s+\lambda+\mu_1+\xi)^{-(j-l+k)} \\
[R^{-(g+j+k-l-n-1)}-R^{-(g+j+k-l+n+1)}]\Big]+\lambda^{-1}\xi[\sum_{i=0}^{n-1}\gamma^{2(n-i)}[(\mu_1+\mu_2)p]^{i}(2\lambda)^{2i+n+1} \\
[(s+\lambda+\mu_1+\mu_2+\xi)+\nu]^{-(2i+n+1)}]+[2(\lambda+\xi+\mu_1)(\lambda+\xi+\mu_2)+\lambda p\mu_1\mu_2 \\
\{(1-p)(\lambda+\xi+\mu_2)+\mu_1\mu_2\}]\sum_{j=0}^{\infty}\sum_{l=0}^{j+1}\sum_{k=0}^{\infty}\sum_{h=0}^{l}\sum_{r=0}^{\infty}(-1)^{l+h}\binom{j+l}{l}\binom{l}{h}\gamma^{-2(j+k)} \\
(s+\lambda+\xi)^{-(k+h-l)}[\{R^{-(g+j+h-l-n)}-R^{-(g+j+h-l+n)}\}-\gamma^{-1}\{R^{-(g+j+h-l-n+1)}-R^{-(g+j+h-l+n+1)}\}] \\
+\lambda\xi(s+\lambda+\mu_2+\xi)^{-2}[\xi\lambda(1-p)(\lambda+\xi+\mu_2)+(\mu_1+\mu_2) \\
\lambda p(\lambda+\xi+\mu_1)]\Bigg[[\sum_{i=0}^{N-2}\gamma^{-2(i+1)}(2\lambda)^{2i-N+2}[(s+\lambda+\mu_1+\mu_2+\xi)+\nu]^{-(2i-N+2)}]- \\
\sum_{i=0}^{N-3}\gamma^{-2(i+3)}(2\lambda)^{2i-N+3}[(s+\lambda+\mu_1+\mu_2+\xi)+\nu]^{-(2i-N+3)}\Bigg]+[(\mu_2-\mu_1)\mu_1\mu_2+ \\
\lambda\{\mu_1\mu_2(1-p)+\mu_1[(\lambda+\xi)(\lambda+\xi+\mu_2)-\lambda p]\}][(s+\lambda+\mu_1+\xi)^{-1}+ \\
4(s+\lambda+\mu_2+\xi)^{2}][\sum_{i=0}^{n-2}\gamma^{-(2n-i+3)}(2\lambda)^{2i-n+2}[(s+\lambda+\mu_1+\mu_2+\xi)+\nu]^{-(2i-n+2)}
\end{array}\right] \tag{40}
$$

$$
\bar{P}_{m,0}(s)=\left[\begin{array}{l}
[2(\mu_1+\mu_2)+\mu_1\mu_2]\sum_{j=0}^{\infty}\sum_{l=0}^{j+1}\sum_{k=0}^{\infty}\sum_{h=0}^{l}\sum_{r=0}^{\infty}\sum_{i=0}^{m-1}(-1)^{i+l+h}\binom{m+j}{j}\binom{j+1}{l}\binom{l}{h}\binom{j+l+r}{r} \\
\binom{m-1}{i}\Big[2^{-(j+1)}\gamma^{-2(j+k+r-l)}(s+\mu_1+\mu_2+\xi)^{-(m-i-l+k+h)}\Big[\{R^{-(g+j+h+i-l-n-1)}-R^{-(g+j+h+i-l+n+1)}\} \\
-2\lambda(\mu_1+\mu_2)\gamma^{-4}\{R^{-(g+j+h+i-l-n)}-R^{-(g+j+h+i-l+n)}\}+[(\lambda+\xi)(\lambda+\xi+\mu_1)+\lambda(\mu_1+\mu_2)] \\
(s+\lambda+\mu_1+\xi)^{-(k+r-l)}\{R^{-(g+j+h+i-l-n)}-R^{-(g+j+h+i-l+n)}\}\Big]\Big]-[(\lambda+\xi+\mu_1)(\lambda+\xi+\mu_2)+\mu_1\mu_2] \\
\Big[\sum_{j=0}^{\infty}\sum_{l=0}^{j+1}\sum_{k=0}^{\infty}\sum_{h=0}^{l}\sum_{r=0}^{\infty}\sum_{i=0}^{m-1}(-1)^{i+l+h}\binom{m+j}{j}\binom{j+l}{l}\binom{l}{h}\binom{j+l+r}{r}\binom{m-1}{i}\gamma^{-(i+k+l+6)}(\mu_1+\mu_2)^{l+h} \\
\lambda^{m-i-l+h}2^{m-i-l+h+r}(s+\mu_1+\mu_2+\xi)^{-(m-i-l+k)}[\{R^{-(g+j+h+i-l-n+1)}-R^{-(g+j+h+i-l+n+1)}\}]-[\xi(\lambda+\xi+\mu_1) \\
(\lambda+\xi+\mu_2)+\mu_1\mu_2]\sum_{j=0}^{\infty}\sum_{l=0}^{j+1}\sum_{k=0}^{\infty}\sum_{i=0}^{N-1}(-1)^{l}\binom{j+1}{l}\binom{j+k}{k}\gamma^{-2(j+k+i+1)}(\mu_1+\mu_2)^{l+i}(2\lambda)^{2(j+k+i)-N+1} \\
[(s+\lambda+\mu_1+\mu_2+\xi)+\nu]^{-[2(j+k+i)-N+1]}+[3\{(\lambda+\xi+\mu_1)(\lambda+\xi+\mu_2)\}+\lambda\mu_1\mu_2(\lambda+\xi+\mu_2) \\
+\xi\lambda(1-p)+\mu_1(1-p)(\lambda+\xi)\mu_1\mu_2]\sum_{j=0}^{\infty}\sum_{l=0}^{j+1}\sum_{k=0}^{\infty}\sum_{h=0}^{l}\sum_{r=0}^{\infty}(-1)^{l+h}\binom{j+1}{l}\binom{j+k}{k}\gamma^{-2(j+k)} \\
(s+\lambda+\xi)^{-(k+h-l)}[\{R^{-(g+j+h-l-n)}-R^{-(g+j+h-l+n)}\}-\gamma^{-1}\{R^{-(g+j+h-l-n+1)}-R^{-(g+j+h-l+n+1)}\}]+ \\
\xi[\lambda\xi(1-p)(\lambda+\xi+\mu_1)+(\mu_1+\mu_2)\mu_1\mu_2(1-p)\lambda+2\mu_2(\lambda+\xi)\mu_1]\Big[(s+\lambda+\mu_1+\xi)^{-3} \\
[\{\sum_{i=0}^{N-2}\gamma^{-2(i+1)}(2\lambda)^{2i-N+2}[(s+\lambda+\mu_1+\mu_2+\xi)+\nu]^{-(2i-N+2)}-\{\sum_{i=0}^{N-1}\gamma^{-2i}(2\lambda)^{2i-N+1} \\
[(s+\lambda+\mu_1+\mu_2+\xi)+\nu]^{-(2i-N+1)}\}]\Big]+[\lambda p(\mu_2-\mu_1)+\mu_1(\mu_2-1)+(\mu_1-1)\mu_2(1-p) \\
+\mu_1(\lambda+\mu_1+\xi)^2+\mu_2(\lambda+\mu_2+\xi)^2]\{3(s+\lambda+\mu_1+\xi)^{-2}+4(s+\lambda+\mu_2+\xi)^{-3}\} \\
[\sum_{i=0}^{n-2}\gamma^{-2(n-i-1)}(2\lambda)^{(2i-n+2)}[(s+\lambda+\mu_1+\mu_2+\xi)+\nu]^{-(2i-n+2)}]
\end{array}\right] \tag{41}
$$

$$
\begin{aligned}
\bar{P}_{m,n}(s)=\frac{\gamma^{2n}}{\lambda}\Bigg[& [2(\mu_1+\mu_2)\gamma^{-1}][\sum_{j=0}^{\infty}\sum_{l=0}^{j+i+1}\sum_{k=0}^{\infty}\sum_{h=0}^{l}\sum_{r=0}^{\infty}\sum_{i=0}^{m-1}\sum_{q=0}^{j-l}\sum_{p1=0}^{h}(-1)^{i+l+p1+q}\binom{m-1}{i}\binom{m+j}{j}\binom{j+i+1}{l} \\
& \binom{l}{h}\binom{j+i+k}{k}\binom{h}{p1}\binom{j+l+r+q}{q}\lambda^{m-i-l+p1}[2(\mu_1+\mu_2)\{\lambda p(\mu_2-\mu_1)+\mu_1\mu_2\}]^{m-i-l-q+p1} \\
& \gamma^{-(2m+j-i-q+1)}(s+\mu_1+\mu_2+\xi)^{-(m-i-l+p1+q)}[\{R^{-(g+j+h+i+p1+1)}-R^{-(g+j+h+i+p1+2)}\}-\gamma^{-4} \\
& \{R^{-(g+j+h+i+p1+q+k)}-R^{-(g+j+h+i+p1+q+k+2)}\}+\{R^{-(g+j+h+i+p1+k+q+r+1)}-R^{-(g+j+h+i+p1+k+q+r+3)}\}]]\Bigg] \\
+\gamma^{2n}\Bigg[& \nu^{-1}\gamma\{R^{(n+1)}-R^{-(n+1)}\}-[2(\mu_1+\mu_2)\{\lambda p(\mu_2-\mu_1)+\mu_1\mu_2\}]\{R^{n}-R^{-n}\}[\sum_{j=0}^{\infty}\sum_{l=0}^{j+i+2}\sum_{k=0}^{\infty}\sum_{h=0}^{l} \\
& \sum_{p1=0}^{h}\sum_{i=0}^{m-1}(-1)^{i+l+h}\binom{m-1}{i}\binom{m+j}{j}\binom{j+i+2}{l}\binom{l}{h}\binom{j+i+k+1}{k}\lambda^{m-i-l+p1}\gamma^{-(2m+j-i-l-p1-1)} \\
& (s+\mu_1+\mu_2+\xi)^{-(m-i-l+p1)}(s+\lambda+\xi)^{-(m-i+k-l)}[R^{-(g+j+h+i+p1+1)}-\gamma^{-1}R^{-(g+j+h+i+p1+2)}]] \\
& -\sum_{j=0}^{\infty}\sum_{l=0}^{j+1}\sum_{k=0}^{\infty}\sum_{i=0}^{N+2}(-1)^{l+i}\binom{j+l}{l}\gamma^{-2(j+k+i+1)}[\lambda^{2}\{p(\mu_2+\mu_1)-\mu_2\}]^{l+i}\{(2\lambda)^{2(j+k+i)-N-2} \\
& [(s+\lambda+\mu_1+\mu_2+\xi)+\nu]^{-[2(j+k+i)-N-2]}\}-[\xi\lambda^{2}\{(\lambda+\xi+\mu_1)+\lambda\mu_1\mu_2(\lambda+\xi+\mu_2)+ \\
& \lambda^{2}\{p(\mu_2+\mu_1)-\mu_2\}\}](s+\lambda+\mu_1+\xi)^{-2}[\{\sum_{i=0}^{N-2}\gamma^{-2(i+1)}(2\lambda)^{2i-N+3} \\
& [(s+\lambda+\mu_1+\mu_2+\xi)+\nu]^{-(2i-N+3)}-\lambda^{-1}(s+\lambda+\mu_2+\xi)^{-2}[\sum_{i=0}^{n-1}\gamma^{-2(n-i)}(2\lambda)^{(2i-n+1)} \\
& [(s+\lambda+\mu_1+\mu_2+\xi)+\nu]^{-(2i-n+1)}-\sum_{i=0}^{n-2}\gamma^{-2(n-i-1)}(2\lambda)^{(2i-n+2)} \\
& [(s+\lambda+\mu_1+\mu_2+\xi)+\nu]^{-(2i-n+2)}]\Bigg] \\
+\Bigg[& [\xi\{(2\mu_2+\mu_1+\xi)(\mu_2+2\mu_1+\xi)(\mu_2+\mu_1)-(\mu_2+\mu_1)\mu_2^{2}(1-p)+(\mu_2+\mu_1)\mu_2\mu_1 \\
& -(\mu_2+\mu_1)\mu_1^{2}p\}]\sum_{j=0}^{\infty}\sum_{l=0}^{j+1}\sum_{k=0}^{\infty}\sum_{h=0}^{l}\sum_{r=0}^{\infty}(-1)^{l+h}\binom{m+j}{j}\binom{j+l}{l}\binom{l}{h}\gamma^{-2(j+k+r)}(s+\lambda+\xi)^{-(k-l+h)} \\
& [\{R^{-(g+j+h-l-n)}-R^{-(g+j+h-l+n)}\}-\gamma^{-1}\{R^{-(g+j+h-l-n+1)}-R^{-(g+j+h-l+n+1)}\}]+\xi\lambda^{-1} \\
& (s+\lambda+\mu_2+\xi)^{-1}[\sum_{i=0}^{N-1}\gamma^{-2i+1}(2\lambda)^{2i-N+2}[(s+\lambda+\mu_1+\mu_2+\xi)+\nu]^{-(2i-N+2)}+[(\mu_2+\mu_1)\lambda p] \\
& (s+\lambda+\mu_1+\xi)^{-1}[\sum_{i=0}^{n-1}\gamma^{-2(n-i)}(2\lambda)^{(2i-n+1)}[(s+\lambda+\mu_1+\mu_2+\xi)+\nu]^{-(2i-n+1)}]]\Bigg]
\end{aligned}
\tag{42}
$$

$$\bar{P}_{m00}(s)=\left[\begin{array}{l}
\xi[(s+\lambda+\beta)(s+\lambda+\beta-\xi)]^{-1}[[2(\mu_1+\mu_2)+\mu_1\mu_2]\sum_{j=0}^{\infty}\sum_{l=0}^{j+1}\sum_{k=0}^{\infty}\sum_{h=0}^{l}\sum_{r=0}^{\infty}\sum_{i=0}^{m-1}(-1)^{j+l+h}\binom{m+j}{j}\binom{j+1}{l}\binom{l}{h}\binom{j+l+r}{r} \\
\binom{m-1}{i}\Big[2^{-(j+1)}\gamma^{-2(j+k+r-l)}(s+\mu_1+\mu_2+\xi)^{-(m-i-l+k+h)}\Big[\{R^{(g+j+h+i-l-n-1)}-R^{(g+j+h+i-l+n+1)}\} \\
-2\lambda(\mu_1+\mu_2)\gamma^{-4}\{R^{(g+j+h+i-l-n)}-R^{(g+j+h+i-l+n)}\}+[(\lambda+\xi)(\lambda+\xi+\mu_1)+\lambda(\mu_1+\mu_2)] \\
(s+\lambda+\mu_1+\xi)^{-(k+r-l)}\{R^{(g+j+h+i-l-n)}-R^{(g+j+h+i-l+n)}\}\Big]\Big]-[(\lambda+\xi+\mu_1)(\lambda+\xi+\mu_2)+\mu_1\mu_2] \\
\Big[\sum_{j=0}^{\infty}\sum_{l=0}^{j+1}\sum_{k=0}^{\infty}\sum_{h=0}^{l}\sum_{r=0}^{\infty}\sum_{i=0}^{m-1}(-1)^{j+l+h}\binom{m+j}{j}\binom{j+l}{l}\binom{l}{h}\binom{j+l+r}{r}\binom{m-1}{i}\gamma^{-(i+k+l+6)}(\mu_1+\mu_2)^{l+h} \\
\lambda^{m-i-l+h}2^{m-i-l+h+r}(s+\mu_1+\mu_2+\xi)^{-(m-i-l+k)}[\{R^{(g+j+h+i-l-n+1)}-R^{(g+j+h+i-l+n+1)}\}]-[\xi(\lambda+\xi+\mu_1) \\
(\lambda+\xi+\mu_2)+\mu_1\mu_2]\sum_{j=0}^{\infty}\sum_{l=0}^{j+1}\sum_{k=0}^{\infty}\sum_{i=0}^{N-1}(-1)^{l}\binom{j+1}{l}\binom{j+k}{k}\gamma^{-2(j+k+i+1)}(\mu_1+\mu_2)^{l+i}(2\lambda)^{2(j+k+i)-N+1} \\
[(s+\lambda+\mu_1+\mu_2+\xi)+v]^{-[2(j+k+i)-N+1]}+[3\{(\lambda+\xi+\mu_1)(\lambda+\xi+\mu_2)\}+\lambda\mu_1\mu_2(\lambda+\xi+\mu_2) \\
+\xi\lambda(1-p)+\mu_1(1-p)(\lambda+\xi)\mu_1\mu_2]\sum_{j=0}^{\infty}\sum_{l=0}^{j+1}\sum_{k=0}^{\infty}\sum_{h=0}^{l}\sum_{r=0}^{\infty}(-1)^{l+h}\binom{j+1}{l}\binom{j+k}{k}\gamma^{-2(j+k)} \\
(s+\lambda+\xi)^{-(k+h-l)}[\{R^{(g+j+h-l-n)}-R^{(g+j+h-l+n)}\}-\gamma^{-1}\{R^{(g+j+h-l-n+1)}-R^{(g+j+h-l+n+1)}\}]+ \\
\xi[\lambda\xi(1-p)(\lambda+\xi+\mu_1)+(\mu_1+\mu_2)\mu_1\mu_2(1-p)\lambda+2\mu_2(\lambda+\xi)\mu_1]\Big[(s+\lambda+\mu_1+\xi)^{-3} \\
[\{\sum_{i=0}^{N-2}\gamma^{-2(i+1)}(2\lambda)^{2i-N+2}[(s+\lambda+\mu_1+\mu_2+\xi)+v]^{-(2i-N+2)}-\{\sum_{i=0}^{N-1}\gamma^{-2i}(2\lambda)^{2i-N+1} \\
[(s+\lambda+\mu_1+\mu_2+\xi)+v]^{-(2i-N+1)}\}]\Big]+[\lambda p(\mu_2-\mu_1)+\mu_1(\mu_2-1)+(\mu_1-1)\mu_2(1-p) \\
+\mu_1(\lambda+\mu_1+\xi)^2+\mu_2(\lambda+\mu_2+\xi)^2]\{3(s+\lambda+\mu_1+\xi)^{-2}+4(s+\lambda+\mu_2+\xi)^{-3}\} \\
[\sum_{i=0}^{n-2}\gamma^{-2(n-i-1)}(2\lambda)^{(2i-n+2)}[(s+\lambda+\mu_1+\mu_2+\xi)+v]^{-(2i-n+2)}]]
\end{array}\right]$$

$$
\frac{\xi}{(s+\lambda+\beta-\xi)}\left[\begin{array}{l}
\frac{\gamma^{2n}}{\lambda}[2(\mu_1+\mu_2)\gamma^{-1}][\sum_{j=0}^{\infty}\sum_{l=0}^{j+i+1}\sum_{k=0}^{\infty}\sum_{h=0}^{l}\sum_{r=0}^{\infty}\sum_{i=0}^{m-1}\sum_{q=0}^{j-l}\sum_{p1=0}^{h}\sum_{n=0}^{N}(-1)^{i+l+p1+q}\binom{m-1}{i}\binom{m+j}{j}\binom{j+i+1}{l} \\
\binom{l}{h}\binom{j+i+k}{k}\binom{h}{p1}\binom{j+l+r+q}{q}\lambda^{m-i-l+p1}[2(\mu_1+\mu_2)\{\lambda p(\mu_2-\mu_1)+\mu_1\mu_2\}]^{m-i-l-q+p1} \\
\gamma^{-(2m+j-i-q+1)}(s+\mu_1+\mu_2+\xi)^{-(m-i-l+p1+q)}[\{R^{-(g+j+h+i+p1+1)}-R^{-(g+j+h+i+p1+2)}\}-\gamma^{-4} \\
\{R^{-(g+j+h+i+p1+q+k)}-R^{-(g+j+h+i+p1+q+k+2)}\}+\{R^{-(g+j+h+i+p1+k+q+r+1)}-R^{-(g+j+h+i+p1+k+q+r+3)}\}]] \\
+\gamma^{2n}v^{-1}\gamma\{R^{(n+1)}-R^{-(n+1)}\}-[2(\mu_1+\mu_2)\{\lambda p(\mu_2-\mu_1)+\mu_1\mu_2\}]\{R^{n}-R^{-n}\}[\sum_{j=0}^{\infty}\sum_{l=0}^{j+i+2}\sum_{k=0}^{\infty}\sum_{n=0}^{N} \\
\sum_{p1=0}^{h}\sum_{h=0}^{l}\sum_{i=0}^{m-1}(-1)^{i+l+h}\binom{m-1}{i}\binom{m+j}{j}\binom{j+i+2}{l}\binom{l}{h}\binom{j+i+k+1}{k}\lambda^{m-i-l+p1}\gamma^{-(2m+j-i-l-p1-1)} \\
(s+\mu_1+\mu_2+\xi)^{-(m-i-l+p1)}(s+\lambda+\xi)^{-(m-i+k-l)}[R^{-(g+j+h+i+p1+1)}-\gamma^{-1}R^{-(g+j+h+i+p1+2)}]] \\
-\sum_{j=0}^{\infty}\sum_{l=0}^{j+1}\sum_{k=0}^{\infty}\sum_{i=0}^{N+2}(-1)^{l+i}\binom{j+l}{l}\gamma^{-2(j+k+i+1)}[\lambda^{2}\{p(\mu_2+\mu_1)-\mu_2\}]^{l+i}\{(2\lambda)^{2(j+k+i)-N-2} \\
[(s+\lambda+\mu_1+\mu_2+\xi)+v]^{-[2(j+k+i)-N-2]}\}-[\xi\lambda^{2}\{(\lambda+\xi+\mu_1)+\lambda\mu_1\mu_2(\lambda+\xi+\mu_2)+ \\
\lambda^{2}\{p(\mu_2+\mu_1)-\mu_2\}\}](s+\lambda+\mu_1+\xi)^{-2}[\{\sum_{i=0}^{N-2}\gamma^{-2(i+1)}(2\lambda)^{2i-N+3} \\
[(s+\lambda+\mu_1+\mu_2+\xi)+v]^{-(2i-N+3)}-\lambda^{-1}(s+\lambda+\mu_2+\xi)^{-2}[\sum_{i=0}^{n-1}\gamma^{-2(n-i)}(2\lambda)^{(2i-n+1)} \\
[(s+\lambda+\mu_1+\mu_2+\xi)+v]^{-(2i-n+1)}-\sum_{i=0}^{n-2}\gamma^{-2(n-i-1)}(2\lambda)^{(2i-n+2)}[(s+\lambda+\mu_1+\mu_2+\xi)+v]^{-(2i-n+2)}] \\
+[\xi\{(2\mu_2+\mu_1+\xi)(\mu_2+2\mu_1+\xi)(\mu_2+\mu_1)-(\mu_2+\mu_1)\mu_2^{2}(1-p)+(\mu_2+\mu_1)\mu_2\mu_1 \\
-(\mu_2+\mu_1)\mu_1^{2}p\}]\sum_{j=0}^{\infty}\sum_{l=0}^{j+1}\sum_{k=0}^{\infty}\sum_{h=0}^{l}\sum_{r=0}^{\infty}\sum_{n=0}^{N}(-1)^{l+h}\binom{m+j}{j}\binom{j+l}{l}\binom{l}{h}\gamma^{-2(j+k+r)}(s+\lambda+\xi)^{-(k-l+h)} \\
[\{R^{-(g+j+h-l-n)}-R^{-(g+j+h-l+n)}\}-\gamma^{-1}\{R^{-(g+j+h-l-n+1)}-R^{-(g+j+h-l+n+1)}\}]+\xi\lambda^{-1} \\
(s+\lambda+\mu_2+\xi)^{-1}[\sum_{i=0}^{N-1}\gamma^{-2i+1}(2\lambda)^{2i-N+2}[(s+\lambda+\mu_1+\mu_2+\xi)+v]^{-(2i-N+2)}+[(\mu_2+\mu_1)\lambda p] \\
(s+\lambda+\mu_1+\xi)^{-1}[\sum_{i=0}^{n-1}\gamma^{-2(n-i)}(2\lambda)^{(2i-n+1)}[(s+\lambda+\mu_1+\mu_2+\xi)+v]^{-(2i-n+1)}]]
\end{array}\right]
$$

(43)

$$\bar{Q}_{m,00}(s)=\frac{\xi}{(s+\lambda+\beta-\xi)}\left[\begin{array}{l}
\frac{\gamma^{2n}}{\lambda}[2(\mu_1+\mu_2)\gamma^{-1}][\sum_{j=0}^{\infty}\sum_{l=0}^{j+i+1}\sum_{k=0}^{\infty}\sum_{h=0}^{l}\sum_{r=0}^{\infty}\sum_{i=0}^{m-1}\sum_{q=0}^{j-l}\sum_{p1=0}^{h}\sum_{n=0}^{N}(-1)^{i+l+p1+q}\binom{m-1}{i}\binom{m+j}{j}\binom{j+i+1}{l} \\
\binom{l}{h}\binom{j+i+k}{k}\binom{h}{p1}\binom{j+l+r+q}{q}\lambda^{m-i-l+p1}[2(\mu_1+\mu_2)\{\lambda p(\mu_2-\mu_1)+\mu_1\mu_2\}]^{m-i-l-q+p1} \\
\gamma^{-(2m+j-i-q+1)}(s+\mu_1+\mu_2+\xi)^{-(m-i-l+p1+q)}[\{R^{-(g+j+h+i+p1+1)}-R^{-(g+j+h+i+p1+2)}\}-\gamma^{-4} \\
\{R^{-(g+j+h+i+p1+q+k)}-R^{-(g+j+h+i+p1+q+k+2)}\}+\{R^{-(g+j+h+i+p1+k+q+r+1)}-R^{-(g+j+h+i+p1+k+q+r+3)}\}]] \\
+\gamma^{2n}v^{-1}\gamma\{R^{(n+1)}-R^{-(n+1)}\}-[2(\mu_1+\mu_2)\{\lambda p(\mu_2-\mu_1)+\mu_1\mu_2\}]\{R^{n}-R^{-n}\}[\sum_{j=0}^{\infty}\sum_{l=0}^{j+i+2}\sum_{k=0}^{\infty}\sum_{n=0}^{N} \\
\sum_{p1=0}^{h}\sum_{h=0}^{l}\sum_{i=0}^{m-1}(-1)^{i+l+h}\binom{m-1}{i}\binom{m+j}{j}\binom{j+i+2}{l}\binom{l}{h}\binom{j+i+k+1}{k}\lambda^{m-i-l+p1} \\
\gamma^{-(2m+j-i-l-p1-1)}(s+\mu_1+\mu_2+\xi)^{-(m-i-l+p1)}(s+\lambda+\xi)^{-(m-i+k-l)}[R^{-(g+j+h+i+p1+1)} \\
-\gamma^{-1}R^{-(g+j+h+i+p1+2)}]]-\sum_{j=0}^{\infty}\sum_{l=0}^{j+1}\sum_{k=0}^{\infty}\sum_{i=0}^{N+2}(-1)^{l+i}\binom{j+l}{l}\gamma^{-2(j+k+i+1)}[\lambda^{2}\{p(\mu_2+\mu_1)-\mu_2\}]^{l+i} \\
\{(2\lambda)^{2(j+k+i)-N-2}[(s+\lambda+\mu_1+\mu_2+\xi)+v]^{-[2(j+k+i)-N-2]}\}-[\xi\lambda^{2}\{(\lambda+\xi+\mu_1)+\lambda\mu_1\mu_2 \\
(\lambda+\xi+\mu_2)+\lambda^{2}\{p(\mu_2+\mu_1)-\mu_2\}\}](s+\lambda+\mu_1+\xi)^{-2}[\{\sum_{i=0}^{N-2}\gamma^{-2(i+1)}(2\lambda)^{2i-N+3} \\
[(s+\lambda+\mu_1+\mu_2+\xi)+v]^{-(2i-N+3)}-\lambda^{-1}(s+\lambda+\mu_2+\xi)^{-2}[\sum_{i=0}^{n-1}\gamma^{-2(n-i)}(2\lambda)^{(2i-n+1)} \\
[(s+\lambda+\mu_1+\mu_2+\xi)+v]^{-(2i-n+1)}-\sum_{i=0}^{n-2}\gamma^{-2(n-i-1)}(2\lambda)^{(2i-n+2)}[(s+\lambda+\mu_1+\mu_2+\xi)+v]^{-(2i-n+2)}] \\
+[\xi\{(2\mu_2+\mu_1+\xi)(\mu_2+2\mu_1+\xi)(\mu_2+\mu_1)-(\mu_2+\mu_1)\mu_2^{2}(1-p)+(\mu_2+\mu_1)\mu_2\mu_1 \\
-(\mu_2+\mu_1)\mu_1^{2}p\}]\sum_{j=0}^{\infty}\sum_{l=0}^{j+1}\sum_{k=0}^{\infty}\sum_{h=0}^{l}\sum_{r=0}^{\infty}\sum_{n=0}^{N}(-1)^{l+h}\binom{m+j}{j}\binom{j+l}{l}\binom{l}{h}\gamma^{-2(j+k+r)}(s+\lambda+\xi)^{-(k-l+h)} \\
[\{R^{-(g+j+h-l-n)}-R^{-(g+j+h-l+n)}\}-\gamma^{-1}\{R^{-(g+j+h-l-n+1)}-R^{-(g+j+h-l+n+1)}\}]+\xi\lambda^{-1} \\
(s+\lambda+\mu_2+\xi)^{-1}[\sum_{i=0}^{N-1}\gamma^{-2i+1}(2\lambda)^{2i-N+2}[(s+\lambda+\mu_1+\mu_2+\xi)+v]^{-(2i-N+2)}+[(\mu_2+\mu_1)\lambda p] \\
(s+\lambda+\mu_1+\xi)^{-1}[\sum_{i=0}^{n-1}\gamma^{-2(n-i)}(2\lambda)^{(2i-n+1)}[(s+\lambda+\mu_1+\mu_2+\xi)+v]^{-(2i-n+1)}]]
\end{array}\right]$$

$$-\xi(s+\lambda+\beta-\xi)^{-1}\left[\begin{array}{l}[2(\mu_1+\mu_2)+\mu_1\mu_2]\sum_{j=0}^{\infty}\sum_{l=0}^{j+1}\sum_{k=0}^{\infty}\sum_{h=0}^{l}\sum_{r=0}^{\infty}\sum_{i=0}^{m-1}(-1)^{i+l+h}\binom{m+j}{j}\binom{j+1}{l}\binom{l}{h}\binom{j+l+r}{r}\binom{m-1}{i}\\ \left[2^{-(j+1)}\gamma^{-2(j+k+r-l)}(s+\mu_1+\mu_2+\xi)^{-(m-i-l+k+h)}\left[\{R^{-(g+j+h+i-l-n-1)}-R^{-(g+j+h+i-l+n+1)}\}\right.\right.\\ -2\lambda(\mu_1+\mu_2)\gamma^{-4}\{R^{-(g+j+h+i-l-n)}-R^{-(g+j+h+i-l+n)}\}+[(\lambda+\xi)(\lambda+\xi+\mu_1)+\lambda(\mu_1+\mu_2)]\\ \left.\left.(s+\lambda+\mu_1+\xi)^{-(k+r-l)}\{R^{-(g+j+h+i-l-n)}-R^{-(g+j+h+i-l+n)}\}\right]\right]-[(\lambda+\xi+\mu_1)(\lambda+\xi+\mu_2)+\mu_1\mu_2]\\ \left[\sum_{j=0}^{\infty}\sum_{l=0}^{j+1}\sum_{k=0}^{\infty}\sum_{h=0}^{l}\sum_{r=0}^{\infty}\sum_{i=0}^{m-1}(-1)^{i+l+h}\binom{m+j}{j}\binom{j+l}{l}\binom{l}{h}\binom{j+l+r}{r}\binom{m-1}{i}\gamma^{-(i+k+l+6)}(\mu_1+\mu_2)^{l+h}\right.\\ \lambda^{m-i-l+h}2^{m-i-l+h+r}(s+\mu_1+\mu_2+\xi)^{-(m-i-l+k)}[\{R^{-(g+j+h+i-l-n+1)}-R^{-(g+j+h+i-l+n+1)}\}]-[\xi(\lambda+\xi+\mu_1)\\ (\lambda+\xi+\mu_2)+\mu_1\mu_2]\sum_{j=0}^{\infty}\sum_{l=0}^{j+1}\sum_{k=0}^{\infty}\sum_{i=0}^{N-1}(-1)^{l}\binom{j+1}{l}\binom{j+k}{k}\gamma^{-2(j+k+i+1)}(\mu_1+\mu_2)^{l+i}(2\lambda)^{2(j+k+i)-N+1}\\ [(s+\lambda+\mu_1+\mu_2+\xi)+v]^{-[2(j+k+i)-N+1]}+[3\{(\lambda+\xi+\mu_1)(\lambda+\xi+\mu_2)\}+\lambda\mu_1\mu_2(\lambda+\xi+\mu_2)\\ +\xi\lambda(1-p)+\mu_1(1-p)(\lambda+\xi)\mu_1\mu_2]\sum_{j=0}^{\infty}\sum_{l=0}^{j+1}\sum_{k=0}^{\infty}\sum_{h=0}^{l}\sum_{r=0}^{\infty}(-1)^{l+h}\binom{j+1}{l}\binom{j+k}{k}\gamma^{-2(j+k)}\\ (s+\lambda+\xi)^{-(k+h-l)}[\{R^{-(g+j+h-l-n)}-R^{-(g+j+h-l+n)}\}-\gamma^{-1}\{R^{-(g+j+h-l-n+1)}-R^{-(g+j+h-l+n+1)}\}]+\\ \xi[\lambda\xi(1-p)(\lambda+\xi+\mu_1)+(\mu_1+\mu_2)\mu_1\mu_2(1-p)\lambda+2\mu_2(\lambda+\xi)\mu_1]\left[(s+\lambda+\mu_1+\xi)^{-3}\right.\\ [\{\sum_{i=0}^{N-2}\gamma^{-2(i+1)}(2\lambda)^{2i-N+2}[(s+\lambda+\mu_1+\mu_2+\xi)+v]^{-(2i-N+2)}-\{\sum_{i=0}^{N-1}\gamma^{-2i}(2\lambda)^{2i-N+1}\\ \left.[(s+\lambda+\mu_1+\mu_2+\xi)+v]^{-(2i-N+1)}\}]\right]+[\lambda p(\mu_2-\mu_1)+\mu_1(\mu_2-1)+(\mu_1-1)\mu_2(1-p)\\ +\mu_1(\lambda+\mu_1+\xi)^2+\mu_2(\lambda+\mu_2+\xi)^2]\{3(s+\lambda+\mu_1+\xi)^{-2}+4(s+\lambda+\mu_2+\xi)^{-3}\}\\ [\sum_{i=0}^{n-2}\gamma^{-2(n-i-1)}(2\lambda)^{(2i-n+2)}[(s+\lambda+\mu_1+\mu_2+\xi)+v]^{-(2i-n+2)}]\end{array}\right] \quad (44)$$

Where

$R=\alpha_1\gamma$, $v=\sqrt{(s+\mu_1+\mu_2+\lambda+\xi)^2-4\lambda(\mu_1+\mu_2)}$, $g=2[l(N+1)+Nk]$ and $\gamma^{-2}=\dfrac{(\mu_1+\mu_2)}{\lambda}$

We are now in a position to complete the solution for the joint distribution of X (t) and Y (t). Taking the Inverse Laplace transform of (38) to (44), using the tables [2, 4, 6], we obtain

$$P_{0,n}(t)=\left[\begin{array}{l}\sum_{j=0}^{\infty}\sum_{l=0}^{j+1}\sum_{k=0}^{\infty}\sum_{h=0}^{l}\sum_{r=0}^{\infty}\sum_{p1=0}^{h}\sum_{q=0}^{j-l}(-1)^{l+h+p1+q}\binom{j+1}{l}\binom{l}{h}\binom{j+l+r}{r}\binom{h}{p1}\binom{j-l}{q}\lambda^{-(j+l-h+p1+q+2)}\\ \gamma^{-2(j+k+h-l+1)}\left[\{2(\mu_1+\mu_2)+\lambda p(\mu_2-\mu_1)+\mu_1\mu_2\right.[[\gamma^{-1}\{I_{(g+j+h-l-n-1)}-I_{(g+j+h-l+n+1)}\}\\ -\gamma^{-2}\{I_{(g+j+h-l-n+1)}-I_{(g+j+h-l+n+1)}\}]e^{-(\lambda+\mu_1+\mu_2+\xi)t}+(\mu_1+\mu_2)^{l+h}\gamma^{-2(k+h+l+1)}\int_0^t\frac{(t-w)^{(j+k-l+h-p1-1)}}{(j+k-l+h-p1-1)!}\\ \left.[I_{(g+j+h+l+k+1)}-\gamma\{I_{(g+j+h+l+k)}-I_{(g+j+h+l+k+2)}\}](2\sqrt{\lambda(\mu_1+\mu_2)}w).e^{-(\lambda p+\xi)w}\right]e^{-(\lambda+\mu_1+\mu_2+\xi)(t-w)}dw+\\ [\lambda^2\xi\{p(\mu_2-\mu_1)+(\lambda+\xi+\mu_1)(\lambda+\xi+\mu_2)(\mu_1+\mu_2)-\lambda\mu_2{}^2(1-p)+\mu_1\mu_2\lambda-\lambda\mu_1{}^2p\}]\\ \left[\begin{array}{l}\sum_{j=0}^{\infty}\sum_{l=0}^{j+1}\sum_{k=0}^{\infty}\sum_{h=0}^{l}\sum_{r=0}^{\infty}(-1)^{l+h}\binom{j+l}{l}\binom{l}{h}\binom{j+l+r}{r}2^{-(j+1)}\lambda^{-(j+k+h-l+2)}\gamma^{-2(j+k+h-l)}[[\{I_{(g+j+h-l-n-1)}\\ -I_{(g+j+h-l+n+1)}\}-\gamma^{-1}\{I_{(g+j+h-l-n)}-I_{(g+j+h-l+n)}\}]e^{-(\lambda+\mu_1+\mu_2+\xi)t}+\int_0^t\frac{(t-w)^{k-l-1}}{(k-l-1)!}[\{I_{(g+j+h-l-n)}-\\ I_{(g+j+h-l+n)}\}-\gamma^{-1}\{I_{(g+j+h-l-n+1)}-I_{(g+j+h-l+n-1)}\}](2\sqrt{\lambda(\mu_1+\mu_2)}w).e^{-(\lambda+\xi)w}]e^{-(\lambda+\mu_1+\mu_2+\xi)(t-w)}dw\end{array}\right]\\ +\xi(\mu_1+\mu_2)[(\lambda+\xi+\mu_1)(\lambda+\xi+\mu_2)+\mu_1\mu_2\{2(\lambda+\xi)+(\mu_1+\mu_2)\}]\int_0^t\frac{(t-w)^2}{2!}\left[[\{\sum_{i=0}^{N-2}\gamma^{-2(i+1)}\right.\\ \left.(2i-N+2)I_{(2i-N+2)}\}-\{\sum_{i=0}^{N-1}\gamma^{-2i}(2i-N+1)I_{(2i-N+1)}\}](2\sqrt{\lambda(\mu_1+\mu_2)}w)w^{-1}.e^{-(\lambda+\mu_1+\xi)w}\right]\\ e^{-(\lambda+\mu_1+\mu_2+\xi)(t-w)}dw]+\gamma^{-2}\lambda^{-1}[-\lambda\{(\lambda+\xi)(\lambda+\xi+\mu_1)(\lambda+\xi+\mu_2)\}-\lambda^2\mu_1p(\lambda+\xi+\mu_2)\\ -\mu_2(1-p)\lambda(\lambda+\xi+\mu_1)]\left[[\{\sum_{i=0}^{n-2}\gamma^{2(n-i-1)}(2i-n+2)I_{(2i-n+2)}\}-3\{\sum_{i=0}^{n-1}\gamma^{2(n-i)}(2i-n+1)\right.\\ \left.I_{(2i-n+1)}\}+\{\sum_{i=0}^{N-2}\gamma^{-2(i+1)}(2i-N+2)I_{(2i-N+2)}\}\right]t^{-1}e^{-(\lambda+\mu_1+\mu_2+\xi)t}+[\sum_{j=0}^{\infty}\sum_{l=0}^{j+1}\sum_{k=0}^{\infty}\sum_{i=0}^{N+2}(-1)^{l+i}\gamma^{-2(j+k+i+1)}\\ \binom{j+l}{l}\binom{N+2}{i}[\{I_{(g+j+l+i-N+2)}-I_{(g+j+l+i-N-2)}\}]+\sum_{j=0}^{\infty}\sum_{l=0}^{j+2}\sum_{k=0}^{\infty}(-1)^{l}\binom{j+2}{l}\binom{j+k+1}{k}\\ \gamma^{-(2j+l-1)}[\{I_{(g+j+i)}-I_{(g+j+3)}\}]-\sum_{i=0}^{N-1}\gamma^{-2i}(2i-N+1)I_{(2i-N+1)}t^{-1}]e^{-(\lambda+\mu_1+\mu_2+\xi)t}\end{array}\right] \quad (45)$$

$$P_{m,1,0}(t)=\left[\begin{array}{l}
\sum_{j=0}^{\infty}\sum_{l=0}^{j+1}\sum_{k=0}^{\infty}\sum_{h=0}^{l}\sum_{r=0}^{\infty}\sum_{p1=0}^{h}\sum_{i=0}^{m-1}(-1)^{i+l+h+p1}\binom{m-1}{i}\binom{m+j}{j}\binom{j+1}{l}\binom{l}{h}\binom{j+l+r+1}{r}\binom{h}{p1} \\
[\lambda p(\mu_1+\mu_2)+\mu_1\mu_2^{\,2}]^{j+k+l}\gamma^{-2(m+j+k-l+1)}2^{-(j+k+p1+2)}\int_0^t\frac{(t-w)^{(m-i+l+k+p1-1)}}{(m-i+l+k+p1-1)!}\Big[\{I_{(g+j+h+i+p1+1)} \\
-I_{(g+j+h+i+p1+2)}\}+\gamma^{-2}\{I_{(g+j+h+i-l-n+1)}-I_{(g+j+h+i-l-n-1)}\}-(2\lambda)^{j+i+k+2}\gamma^{-6}[\lambda(\mu_1+\mu_2)+\lambda(1-p) \\
\mu_1\mu_2+\mu_2\{(\lambda+\xi)(\lambda+\xi+\mu_2)-\lambda p\mu_1\}][I_{(g+j+h+i+p1-l-n)}-I_{(g+j+h+i+p1-l+n)}]\Big](2\sqrt{\lambda(\mu_1+\mu_2)}w) \\
e^{-(\mu_1+\mu_2+\xi)(t-w)}dw+[\lambda\xi p(\lambda+\xi+\mu_2)+\lambda^2\{(\mu_1+\mu_2)p-\mu_2\}+\mu_1\mu_2(1-p)\lambda+\mu_2\{(\lambda+\xi) \\
(\lambda+\xi+\mu_2)-\lambda p\mu_1\mu_2\}]\left[\sum_{j=0}^{\infty}\sum_{l=0}^{j+i+1}\sum_{k=0}^{\infty}\sum_{i=0}^{m-1}(-1)^{i+l}\binom{j+i+l}{l}\binom{m-1}{i}\binom{j+k}{k}\lambda^{m-i+1}\mu_1^{-(l+k)}\gamma^{-(2m+j-i-1)}\right. \\
\int_0^t\frac{(t-w)^{(m-i+l-1)}}{(m-i+l-1)!}[\{I_{(g+j+l+n-1)}-I_{(g+j+l-n+1)}\}(2\sqrt{\lambda(\mu_1+\mu_2)}w)+[\lambda^2\xi\{(\mu_1+\mu_2)p-\mu_2\}]\int_0^w\frac{(w-u)^2}{2!} \\
\left.[I_{(g+j+l+k-n-1)}-I_{(g+j+l+k-n+1)}](2\sqrt{\lambda(\mu_1+\mu_2)}u)e^{-(\lambda+\xi+\mu_2)u}du\right]e^{-(\mu_1+\mu_2+\xi)(t-w)}dw-\sum_{j=0}^{\infty}\sum_{l=0}^{j+1}\sum_{k=0}^{\infty}\sum_{i=0}^{N+2}(-1)^{l} \\
\binom{j+l}{l}[\lambda^2[(\mu_1+\mu_2)p-\mu_2]^{l+i}\Big[(2(j+k+i)-N-2)I_{[2(j+k+i)-N-2]}t^{-1}e^{-(\lambda+\mu_1+\mu_2+\xi)t}+\Big[\int_0^t\frac{(t-w)^{j-l+i-1}}{(j-l+i-1)!} \\
[I_{(g+j+k-l-n-1)}-I_{(g+j+k-l+n-1)}](2\sqrt{\lambda(\mu_1+\mu_2)}w)\Big]e^{-(\lambda+\mu_1+\xi)(t-w)}dw+[\lambda\xi[(\mu_1+\mu_2)p-\mu_2]]\int_0^t\frac{(t-w)^2}{2!} \\
[\sum_{i=0}^{n-1}\gamma^{2(n-i)}(2i+n+1)I_{(2i+n+1)}(2\sqrt{\lambda(\mu_1+\mu_2)}w).w^{-1}e^{-(\lambda+\xi+\mu_1)w}]e^{-(\lambda+\xi+\mu_1+\mu_2)(t-w)}dw+[2(\lambda+\xi+\mu_1) \\
(\lambda+\xi+\mu_2)+\lambda p\mu_1\mu_2(\lambda+\xi+\mu_2)+\lambda^3[(\mu_1+\mu_2)p-\mu_2]]\sum_{j=0}^{\infty}\sum_{l=0}^{j+1}\sum_{k=0}^{\infty}\sum_{h=0}^{l}\sum_{r=0}^{\infty}(-1)^{l+h}\binom{j+1+l}{l} \\
\binom{m+j}{j}\binom{l}{h}\gamma^{-2(j+k+r)}\int_0^t\frac{(t-w)^{(k+h-l-1)}}{(k+h-l-1)!}[\{I_{(g+j+h-l-n)}-I_{(g+j+h-l+n)}\}-\gamma^{-1}\{I_{(g+j+h-l-n+1)}-I_{(g+j+h-l+n+1)}\}] \\
(2\sqrt{\lambda(\mu_1+\mu_2)}w).e^{-(\lambda+\xi)(t-w)}dw+\lambda\xi\int_0^t[\frac{(t-w)^2}{2!}[\{\sum_{i=0}^{N+1}\gamma^{-2(i+2)}(2i-N-1)I_{(2i-N-1)}\}+\{\sum_{i=0}^{N-2}\gamma^{-2(i+1)} \\
(2i-N+2)I_{(2i-N+2)}\}](2\sqrt{\lambda(\mu_1+\mu_2)}w).w^{-1}e^{-\xi w}]e^{-(\lambda+\xi+\mu_1+\mu_2)(t-w)}dw
\end{array}\right] \quad (46)$$

$$
P_{m,0,1}(t)=\left[\begin{array}{l}
\sum_{j=0}^{\infty}\sum_{l=0}^{j+1}\sum_{k=0}^{\infty}\sum_{h=0}^{l}\sum_{r=0}^{\infty}\sum_{p1=0}^{h}\sum_{i=0}^{m-1}(-1)^{i+l+h+p1}\binom{j+m}{j}\binom{j+1}{l}\binom{l}{h}\binom{j+l+r+1}{r}\binom{h}{p1}\binom{m-1}{i} \\
[\lambda(\mu_1+\mu_2)+\mu_1\mu_2]^{j+k+l}\gamma^{-2(m+j+k+r-l+1)}2^{-(j+k+p1+1)}\int_0^t\frac{(t-w)^{(m-i+l+k+p1-1)}}{(m-i+l+k+p1-1)!}\Big[\{I_{(g+j+h+i+p1+1)}- \\
\gamma^{-2}I_{(g+j+h+i+p1+2)}\}+\{I_{(g+j+h+i-l-n-1)}-I_{(g+j+h+i-l-n+1)}\}-(2\lambda)^{i+k+2}\gamma^{-4}[\lambda(\mu_1+\mu_2)+(1-p)\mu_1\mu_2 \\
+\mu_1\lambda\{(\lambda+\xi)(\lambda+\xi+\mu_1)-\lambda p\mu_1\}][I_{(g+j+h+i+p1-l-n)}-I_{(g+j+h+i+p1-l+n)}]\Big](2\sqrt{\lambda(\mu_1+\mu_2)}w) \\
e^{-(\mu_1+\mu_2+\xi)(t-w)}dw+[\lambda\xi p(\lambda+\xi+\mu_2)+\mu_1\mu_2(1-p)\lambda+\mu_2\{(\lambda+\xi)(\lambda+\xi+\mu_2)-\lambda(1-p)\mu_2\}] \\
\Bigg[\sum_{j=0}^{\infty}\sum_{l=0}^{j+i+1}\sum_{k=0}^{\infty}\sum_{i=0}^{m-1}(-1)^{i+l}\binom{j+i+1}{l}\binom{m-1}{i}\binom{j+i+k}{k}\lambda^{m-i+1}\mu_2^{-(l+k)}\gamma^{-(2m+j-i-1)}\int_0^t\frac{(t-w)^{(m-i+l-1)}}{(m-i+l-1)!} \\
[\{I_{(g+j+l+n-1)}-I_{(g+j+l-n+1)}\}(2\sqrt{\lambda(\mu_1+\mu_2)}w)+\xi\int_0^w\frac{(w-u)^2}{2!}[\{I_{(g+j+l+k-n-1)}-I_{(g+j+l+k+n-1)}\} \\
(2\sqrt{\lambda(\mu_1+\mu_2)}u)]e^{-(\lambda+\xi+\mu_1)u}du\Bigg]e^{-(\mu_1+\mu_2+\xi)(t-w)}dw-\sum_{j=0}^{\infty}\sum_{l=0}^{j+1}\sum_{k=0}^{\infty}\sum_{i=0}^{N-1}(-1)^{i+l}\binom{j+1}{l}\binom{N-1}{i}\binom{N+j}{j} \\
\gamma^{-2(j+k+i+1)}[\lambda[(\mu_1+\mu_2)p]^{l+i}\Bigg[\{(2(j+k+i)-N+1)I_{[2(j+k+i)-N+1]}t^{-1}e^{-(\lambda+\mu_1+\mu_2+\xi)t}\}+\int_0^t\frac{(t-w)^{(j-l+k-1)}}{(j-l+k-1)!} \\
[\{I_{(g+j+k-l-n-1)}-I_{(g+j+k-l+n+1)}\}(2\sqrt{\lambda(\mu_1+\mu_2)}w)]e^{-(\lambda+\xi+\mu_1)(t-w)}dw\Bigg]+\lambda^{-1}\xi[\sum_{i=0}^{n-1}\gamma^{2(n-i)}[(\mu_1+\mu_2)p]^i \\
(2i+n+1)I_{(2i+n+1)}t^{-1}e^{-(\lambda+\mu_1+\mu_2+\xi)t}]+[2(\lambda+\xi+\mu_1)(\lambda+\xi+\mu_2)+\lambda p\mu_1\mu_2\{(1-p)(\lambda+\xi+\mu_2) \\
+\mu_1\mu_2\}]\sum_{j=0}^{\infty}\sum_{l=0}^{j+1}\sum_{k=0}^{\infty}\sum_{h=0}^{l}\sum_{r=0}^{\infty}(-1)^{l+h}\binom{j+l}{l}\binom{l}{h}\gamma^{-2(j+k)}\int_0^t\frac{(t-w)^{(k+h-l-1)}}{(k+h-l-1)!}[\{I_{(g+j+h-l-n)}-I_{(g+j+h-l+n)}\} \\
-\gamma^{-1}\{I_{(g+j+h-l-n+1)}-I_{(g+j+h-l+n+1)}\}](2\sqrt{\lambda(\mu_1+\mu_2)}w)]e^{-(\lambda+\xi)(t-w)}dw+\lambda\xi[\xi\lambda(1-p)(\lambda+\xi+\mu_2) \\
+(\mu_1+\mu_2)\lambda p(\lambda+\xi+\mu_1)]\int_0^t\frac{(t-w)^2}{2!}\Bigg[[\sum_{i=0}^{N-2}\gamma^{-2(i+1)}(2i-N+2)I_{(2i-N+2)}-\sum_{i=0}^{N-3}\gamma^{-2(i+3)}(2i-N+3) \\
I_{(2i-N+3)}](2\sqrt{\lambda(\mu_1+\mu_2)}w)]w^{-1}e^{-(\lambda+\mu_1+\mu_2+\xi)w}\Bigg]e^{-(\lambda+\xi+\mu_2)(t-w)}dw+[(\mu_2-\mu_1)\mu_1\mu_2+\lambda\{\mu_1\mu_2(1-p) \\
+\mu_1[(\lambda+\xi)(\lambda+\xi+\mu_2)-\lambda p]\}][\int_0^t e^{-(\lambda+\xi+\mu_1)(t-w)}+4\int_0^t\frac{(t-w)}{1!}e^{-(\lambda+\xi+\mu_2)(t-w)}] \\
[\{\sum_{i=0}^{n-2}\gamma^{-(2n-i+3)}(2i-n+2)I_{(2i-n+2)}\}(2\sqrt{\lambda(\mu_1+\mu_2)}w)w^{-1}e^{-(\lambda+\mu_1+\mu_2+\xi)w}]dw
\end{array}\right]
\tag{47}
$$

$$
P_{m,0}(t)=\left[\begin{array}{l}
[2(\mu_1+\mu_2)+\mu_1\mu_2]\sum_{j=0}^{\infty}\sum_{l=0}^{j+1}\sum_{k=0}^{\infty}\sum_{h=0}^{l}\sum_{r=0}^{\infty}\sum_{i=0}^{m-1}(-1)^{i+l+h}\binom{m+j}{j}\binom{j+1}{l}\binom{l}{h}\binom{j+l+r}{r}\\
\binom{m-1}{i}\Big[2^{-(j+1)}\gamma^{-2(j+k+r-l)}\int_0^t\frac{(t-w)^{(m-i-l+k+h-1)}}{(m-i-l+k+h-1)!}\Big[[\{I_{(g+j+h+i-l-n-1)}-I_{(g+j+h+i-l+n+1)}\}\\
-2\lambda(\mu_1+\mu_2)\gamma^{-4}\{I_{(g+j+h+i-l-n)}-I_{(g+j+h+i-l+n)}\}](2\sqrt{\lambda(\mu_1+\mu_2)}w)+[(\lambda+\xi)(\lambda+\xi+\\
\mu_1)+\lambda(\mu_1+\mu_2)][\int_0^w\frac{(w-u)^{(k+r-l-1)}}{(k+r-l-1)!}\{I_{(g+j+h+i-l-n)}-I_{(g+j+h+i-l+n)}\}(2\sqrt{\lambda(\mu_1+\mu_2)}u)]\\
e^{-(\lambda+\xi+\mu_1)u}\Big]e^{-(\mu_1+\mu_2+\xi)(t-w)}dw\Big]-[(\lambda+\xi+\mu_1)(\lambda+\xi+\mu_2)+\mu_1\mu_2]\\
\Big[\sum_{j=0}^{\infty}\sum_{l=0}^{j+1}\sum_{k=0}^{\infty}\sum_{h=0}^{l}\sum_{r=0}^{\infty}\sum_{i=0}^{m-1}(-1)^{i+l+h}\binom{m+j}{j}\binom{j+l}{l}\binom{l}{h}\binom{j+l+r}{r}\binom{m-1}{i}\gamma^{-(i+k+l+6)}(\mu_1+\mu_2)^{l+h}\\
\lambda^{m-i-l+h}2^{m-i-l+h+r}\int_0^t\frac{(t-w)^{(m-i-l+k-1)}}{(m-i-l+k-1)!}[\{I_{(g+j+h+i-l-n+1)}-I_{(g+j+h+i-l+n+1)}\}(2\sqrt{\lambda(\mu_1+\mu_2)}w)]\\
e^{-(\mu_1+\mu_2+\xi)(t-w)}dw-[\xi(\lambda+\xi+\mu_1)(\lambda+\xi+\mu_2)+\mu_1\mu_2]\sum_{j=0}^{\infty}\sum_{l=0}^{j+1}\sum_{k=0}^{\infty}\sum_{i=0}^{N-1}(-1)^{l}\binom{j+1}{l}\binom{j+k}{k}\\
\gamma^{-2(j+k+i+1)}(\mu_1+\mu_2)^{l+i}(2(j+k+i)-N+1)I_{[2(j+k+i)-N+1]}t^{-1}e^{-(\lambda+\mu_1+\mu_2+\xi)t}+[3\{(\lambda+\xi+\mu_1)\\
(\lambda+\xi+\mu_2)\}+\lambda\mu_1\mu_2(\lambda+\xi+\mu_2)+\xi\lambda(1-p)+\mu_1(1-p)(\lambda+\xi)\mu_1\mu_2]\\
\sum_{j=0}^{\infty}\sum_{l=0}^{j+1}\sum_{k=0}^{\infty}\sum_{h=0}^{l}\sum_{r=0}^{\infty}(-1)^{l+h}\binom{j+1}{l}\binom{j+k}{k}\gamma^{-2(j+k)}\int_0^t\frac{(t-w)^{(k+h-l-1)}}{(k+h-l-1)!}[\{I_{(g+j+h-l-n)}-I_{(g+j+h-l+n)}\}\\
-\gamma^{-1}\{I_{(g+j+h-l-n+1)}-I_{(g+j+h-l+n+1)}\}](2\sqrt{\lambda(\mu_1+\mu_2)}w)e^{-(\lambda+\xi)w}dw+\xi[\lambda\xi(1-p)(\lambda+\xi+\mu_1)\\
+(\mu_1+\mu_2)\mu_1\mu_2(1-p)\lambda+2\mu_2(\lambda+\xi)\mu_1]\Big[\int_0^t\frac{(t-w)^2}{2!}[\{\sum_{i=0}^{N-2}\gamma^{-2(i+1)}(2i-N+2)I_{(2i-N+2)}\}\\
-\{\sum_{i=0}^{N-1}\gamma^{-2i}(2i-N+1)I_{(2i-N+1)}\}](2\sqrt{\lambda(\mu_1+\mu_2)}w)w^{-1}e^{-(\lambda+\mu_1+\xi)w}\Big]e^{-(\lambda+\mu_1+\mu_2+\xi)(t-w)}dw+\\
[\lambda p(\mu_2-\mu_1)+\mu_1(\mu_2-1)+(\mu_1-1)\mu_2(1-p)+\mu_1(\lambda+\mu_1+\xi)^2+\mu_2(\lambda+\mu_2+\xi)^2]\\
[\{3\int_0^t\frac{(t-w)}{1!}e^{-(\lambda+\xi+\mu_1)(t-w)}+4\int_0^t\frac{(t-w)^2}{2!}e^{-(\lambda+\xi+\mu_2)(t-w)}\}dw][\sum_{i=0}^{n-2}\gamma^{-2(n-i-1)}(2i-n+2)I_{(2i-n+2)}]
\end{array}\right]
\tag{48}
$$

$$
\begin{aligned}
P_{m,n}(t) = \frac{\gamma^{2n}}{\lambda} \Bigg[& [2(\mu_1+\mu_2)\gamma^{-1}][\sum_{j=0}^{\infty}\sum_{l=0}^{j+i+1}\sum_{k=0}^{\infty}\sum_{h=0}^{l}\sum_{r=0}^{\infty}\sum_{i=0}^{m-1}\sum_{q=0}^{j-l}\sum_{p1=0}^{h}(-1)^{i+l+p1+q}\binom{m-1}{i}\binom{m+j}{j}\binom{j+i+1}{l} \\
& \binom{l}{h}\binom{j+i+k}{k}\binom{h}{p1}\binom{j+l+r+q}{q}\lambda^{m-i-l+p1}[2(\mu_1+\mu_2)\{\lambda p(\mu_2-\mu_1)+\mu_1\mu_2\}]^{m-i-l-q+p1} \\
& \gamma^{-(2m+j-i-q+1)}\int_0^t \frac{(t-w)^{(m-i-l+p1+q-1)}}{(m-i-l+p1+q-1)!}[\{I_{(g+j+h+i+p1+1)}-I_{(g+j+h+i+p1+2)}\}-\gamma^{-4} \\
& \{I_{(g+j+h+i+p1+q+k)}-I_{(g+j+h+i+p1+q+k+2)}\}+\{I_{(g+j+h+i+p1+k+q+r+1)}-I_{(g+j+h+i+p1+k+q+r+3)}\} \\
&](2\sqrt{\lambda(\mu_2+\mu_1)w})]e^{-(\mu_1+\mu_2+\xi)(t-w)}dw \Bigg]
\end{aligned}
$$

$$
\begin{aligned}
+ \Bigg[& [\sum_{j=0}^{\infty}\sum_{l=0}^{j+i+2}\sum_{k=0}^{\infty}\sum_{h=0}^{l}\sum_{p1=0}^{h}\sum_{i=0}^{m-1}(-1)^{i+l+h}\binom{m-1}{i}\binom{m+j}{j}\binom{j+i+2}{l}\binom{l}{h}\binom{j+i+k+1}{k}\lambda^{m-i-l+p1} \\
& \gamma^{-(2m+j-i-l-2n-1)}\int_0^t \frac{(t-w)^{(m-i-l+p1-1)}}{(m-i-l+p1-1)!}\Bigg[\int_0^w \frac{(w-u)^{(m-i-l+k-1)}}{(m-i-l+k-1)!}[\gamma\{(g+j+h+i+p1-n) \\
& I_{(g+j+h+i+p1-n)}-(g+j+h+i+p1+n+2)I_{(g+j+h+i+p1+n+2)}\}-\gamma^{-1}\{(g+j+h+i+p1-n+1) \\
& I_{(g+j+h+i+p1-n+1)}-(g+j+h+i+p1+n+3)I_{(g+j+h+i+p1+n+3)}\}]-[2(\mu_1+\mu_2)\{\lambda p(\mu_2-\mu_1) \\
& +\mu_1\mu_2\}][\{(g+j+h+i+p1-n+1)I_{(g+j+h+i+p1-n+1)}-(g+j+h+i+p1+n+1) \\
& I_{(g+j+h+i+p1+n+1)}\}-\gamma^{-1}\{(g+j+h+i+p1+n+2)I_{(g+j+h+i+p1+n+2)}- \\
& (g+j+h+i+p1-n+2)I_{(g+j+h+i+p1-n+2)}\}](2\sqrt{\lambda(\mu_2+\mu_1)}u)^{-1}e^{-(\lambda+\xi)(w-u)}du]e^{-(\mu_1+\mu_2+\xi)(t-w)}dw \\
& -\sum_{j=0}^{\infty}\sum_{l=0}^{j+1}\sum_{k=0}^{\infty}\sum_{i=0}^{N+2}(-1)^{l+i}\binom{j+l}{l}\gamma^{-2(j+k+i+1)}[\lambda^2\{p(\mu_2+\mu_1)-\mu_2\}]^{l+i}\{(2(j+k+i)-N-2) \\
& I_{[2(j+k+i)-N-2]}t^{-1}e^{-(\lambda+\mu_1+\mu_2+\xi)t}\}-[\xi\lambda^2\{(\lambda+\xi+\mu_1)^2+\lambda\mu_1\mu_2(\lambda+\xi+\mu_2)+\lambda^2\{p(\mu_2+\mu_1) \\
& -\mu_2\}\}]\int_0^t \frac{(t-w)}{1!}[\{\sum_{i=0}^{N-2}\gamma^{-2(i+1)}(2i-N+3)I_{(2i-N+3)}\}(2\sqrt{\lambda(\mu_2+\mu_1)}w).w^{-1}]e^{-(\lambda+\mu_1+\mu_2+\xi)w} \\
& -\lambda^{-1}\int_0^w \frac{(w-u)}{1!}[\{\sum_{i=0}^{n-1}\gamma^{-2(n-i)}(2i-n+1)I_{(2i-n+1)}-\sum_{i=0}^{n-2}\gamma^{-2(n-i-1)}(2i-n+2)I_{(2i-n+2)}\}(2\sqrt{\lambda(\mu_2+\mu_1)}u) \\
& .u^{-1}.e^{-(\lambda+\mu_1+\mu_2+\xi)u}]e^{-(\lambda+\mu_2+\xi)(w-u)}du]e^{-(\lambda+\mu_1+\xi)(t-w)}dw \Bigg]
\end{aligned}
$$

$$
\begin{aligned}
+ \Bigg[& [\xi\{(2\mu_2+\mu_1+\xi)(\mu_2+2\mu_1+\xi)(\mu_2+\mu_1)-(\mu_2+\mu_1)\mu_2{}^2(1-p)+(\mu_2+\mu_1)\mu_2\mu_1 \\
& -(\mu_2+\mu_1)\mu_1{}^2 p\}]\sum_{j=0}^{\infty}\sum_{l=0}^{j+1}\sum_{k=0}^{\infty}\sum_{h=0}^{l}\sum_{r=0}^{\infty}(-1)^{l+h}\binom{m+j}{j}\binom{j+l}{l}\binom{l}{h}\gamma^{-2(j+k+r)}\int_0^t \frac{(t-w)^{(k-l+h-1)}}{(k-l+h-1)!} \\
& [\{I_{(g+j+h-l-n)}-I_{(g+j+h-l+n)}\}-\gamma^{-1}\{I_{(g+j+h-l-n+1)}-I_{(g+j+h-l+n+1)}\}](2\sqrt{\lambda(\mu_2+\mu_1)}w) \\
& e^{-(\lambda+\mu_1+\mu_2+\xi)(t-w)}+\xi\lambda^{-1}\int_0^t[\{\sum_{i=0}^{N-1}\gamma^{-2i+1}(2i-N+2)I_{(2i-N+2)}\}(2\sqrt{\lambda(\mu_2+\mu_1)}w)w^{-1}e^{-(\lambda+\mu_1+\mu_2+\xi)w} \\
& +[(\mu_2+\mu_1)\lambda p]\int_0^w[\sum_{i=0}^{n-1}\gamma^{-2(n-i)}(2i-n+1)I_{(2i-n+1)}](2\sqrt{\lambda(\mu_2+\mu_1)}u)u^{-1}.e^{-(\lambda+\mu_1+\mu_2+\xi)u}] \\
& e^{-(\lambda+\mu_1+\xi)(w-u)}du]e^{-(\lambda+\mu_2+\xi)(t-w)}dw] \Bigg]
\end{aligned}
$$

(49)

$$
P_{m,00}(t)=\left[\begin{aligned}
&[2(\mu_1+\mu_2)+\mu_1\mu_2]\sum_{j=0}^{\infty}\sum_{l=0}^{j+1}\sum_{k=0}^{\infty}\sum_{h=0}^{l}\sum_{r=0}^{\infty}\sum_{i=0}^{m-1}(-1)^{i+l+h}\binom{m+j}{j}\binom{j+1}{l}\binom{l}{h}\binom{j+l+r}{r}\\
&\binom{m-1}{i}\Big[2^{-(j+1)}\gamma^{-2(j+k+r-l)}\int_0^t\frac{(t-w)^{(m-i-l+k+h-1)}}{(m-i-l+k+h-1)!}\Big[\{(\lambda+\beta)e^{-(\lambda+\beta-\xi)w}-(\lambda+\beta-\xi)\\
&e^{-(\lambda+\beta-\xi)w}\}\Big[[\{I_{(g+j+h+i-l-n-1)}-I_{(g+j+h+i-l+n+1)}\}\\
&-2\lambda(\mu_1+\mu_2)\gamma^{-4}\{I_{(g+j+h+i-l-n)}-I_{(g+j+h+i-l+n)}\}](2\sqrt{\lambda(\mu_1+\mu_2)}w)+[(\lambda+\xi)(\lambda+\xi+\\
&\mu_1)+\lambda(\mu_1+\mu_2)][\int_0^w\frac{(w-u)^{(k+r-l-1)}}{(k+r-l-1)!}\{I_{(g+j+h+i-l-n)}-I_{(g+j+h+i-l+n)}\}(2\sqrt{\lambda(\mu_1+\mu_2)}u)]\\
&e^{-(\lambda+\xi+\mu_1)u}\Big]e^{-(\mu_1+\mu_2+\xi)(t-w)}dw\Big]-[(\lambda+\xi+\mu_1)(\lambda+\xi+\mu_2)+\mu_1\mu_2]\\
&\Big[\sum_{j=0}^{\infty}\sum_{l=0}^{j+1}\sum_{k=0}^{\infty}\sum_{h=0}^{l}\sum_{r=0}^{\infty}\sum_{i=0}^{m-1}(-1)^{i+l+h}\binom{m+j}{j}\binom{j+l}{l}\binom{l}{h}\binom{j+l+r}{r}\binom{m-1}{i}\gamma^{-(i+k+l+6)}(\mu_1+\mu_2)^{l+h}\\
&\lambda^{m-i-l+h}2^{m-i-l+h+r}\int_0^t\frac{(t-w)^{(m-i-l+k-1)}}{(m-i-l+k-1)!}[\{I_{(g+j+h+i-l-n+1)}-I_{(g+j+h+i-l+n+1)}\}(2\sqrt{\lambda(\mu_1+\mu_2)}w)]\\
&e^{-(\mu_1+\mu_2+\xi)(t-w)}dw-[\xi(\lambda+\xi+\mu_1)(\lambda+\xi+\mu_2)+\mu_1\mu_2]\sum_{j=0}^{\infty}\sum_{l=0}^{j+1}\sum_{k=0}^{\infty}\sum_{i=0}^{N-1}(-1)^{l}\binom{j+1}{l}\binom{j+k}{k}\\
&\gamma^{-2(j+k+i+1)}(\mu_1+\mu_2)^{l+i}(2(j+k+i)-N+1)I_{[2(j+k+i)-N+1]}t^{-1}e^{-(\lambda+\mu_1+\mu_2+\xi)t}+[3\{(\lambda+\xi+\mu_1)\\
&(\lambda+\xi+\mu_2)\}+\lambda\mu_1\mu_2(\lambda+\xi+\mu_2)+\xi\lambda(1-p)+\mu_1(1-p)(\lambda+\xi)\mu_1\mu_2]\\
&\sum_{j=0}^{\infty}\sum_{l=0}^{j+1}\sum_{k=0}^{\infty}\sum_{h=0}^{l}\sum_{r=0}^{\infty}(-1)^{l+h}\binom{j+1}{l}\binom{j+k}{k}\gamma^{-2(j+k)}\int_0^t\frac{(t-w)^{(k+h-l-1)}}{(k+h-l-1)!}[\{I_{(g+j+h-l-n)}-I_{(g+j+h-l+n)}\}\\
&-\gamma^{-1}\{I_{(g+j+h-l-n+1)}-I_{(g+j+h-l+n+1)}\}](2\sqrt{\lambda(\mu_1+\mu_2)}w)e^{-(\lambda+\xi)w}dw+\xi[\lambda\xi(1-p)(\lambda+\xi+\mu_1)\\
&+(\mu_1+\mu_2)\mu_1\mu_2(1-p)\lambda+2\mu_2(\lambda+\xi)\mu_1]\Big[\int_0^t\frac{(t-w)^2}{2!}[\{\sum_{i=0}^{N-2}\gamma^{-2(i+1)}(2i-N+2)I_{(2i-N+2)}\}\\
&-\{\sum_{i=0}^{N-1}\gamma^{-2i}(2i-N+1)I_{(2i-N+1)}\}](2\sqrt{\lambda(\mu_1+\mu_2)}w)w^{-1}e^{-(\lambda+\mu_1+\xi)w}\Big]e^{-(\lambda+\mu_1+\mu_2+\xi)(t-w)}dw+\\
&[\lambda p(\mu_2-\mu_1)+\mu_1(\mu_2-1)+(\mu_1-1)\mu_2(1-p)+\mu_1(\lambda+\mu_1+\xi)^2+\mu_2(\lambda+\mu_2+\xi)^2]\\
&[\{3\int_0^t\frac{(t-w)}{1!}e^{-(\lambda+\xi+\mu_1)(t-w)}+4\int_0^t\frac{(t-w)^2}{2!}e^{-(\lambda+\xi+\mu_2)(t-w)}\}dw][\sum_{i=0}^{n-2}\gamma^{-2(n-i-1)}(2i-n+2)I_{(2i-n+2)}]
\end{aligned}\right]
$$

$$
-\xi\left[\begin{array}{l}
\frac{\gamma^{2n}}{\lambda}[2(\mu_1+\mu_2)\gamma^{-1}][\sum_{j=0}^{\infty}\sum_{l=0}^{j+i+1}\sum_{k=0}^{\infty}\sum_{h=0}^{l}\sum_{r=0}^{\infty}\sum_{i=0}^{m-1}\sum_{q=0}^{j-l}\sum_{p1=0}^{h}\sum_{n=0}^{N}(-1)^{i+l+p1+q}\binom{m-1}{i}\binom{m+j}{j}\binom{j+i+1}{l} \\
\binom{l}{h}\binom{j+i+k}{k}\binom{h}{p1}\binom{j+l+r+q}{q}\lambda^{m-i-l+p1}[2(\mu_1+\mu_2)\{\lambda p(\mu_2-\mu_1)+\mu_1\mu_2\}]^{m-i-l-q+p1} \\
\gamma^{-(2m+j-i-q+1)}\int_0^t\frac{(t-w)^{(m-i-l+p1+q-1)}}{(m-i-l+p1+q-1)!}[\{I_{(g+j+h+i+p1+1)}-I_{(g+j+h+i+p1+2)}\}-\gamma^{-4} \\
\{I_{(g+j+h+i+p1+q+k)}-I_{(g+j+h+i+p1+q+k+2)}\}+\{I_{(g+j+h+i+p1+k+q+r+1)}-I_{(g+j+h+i+p1+k+q+r+3)}\} \\
](2\sqrt{\lambda(\mu_2+\mu_1)}w)e^{-(\lambda+\beta-\xi)w}]e^{-(\mu_1+\mu_2+\xi)(t-w)}dw+[\sum_{j=0}^{\infty}\sum_{l=0}^{j+i+2}\sum_{k=0}^{\infty}\sum_{h=0}^{l}\sum_{p1=0}^{h}\sum_{i=0}^{m-1}\sum_{n=0}^{N}(-1)^{i+l+h}\binom{m-1}{i}\binom{m+j}{j} \\
\binom{j+i+2}{l}\binom{l}{h}\binom{j+i+k+1}{k}\lambda^{m-i-l+p1}\gamma^{-(2m+j-i-l-2n-1)}\int_0^t\frac{(t-w)^{(m-i-l+p1-1)}}{(m-i-l+p1-1)!}\Big[\int_0^w\frac{(w-u)^{(m-i-l+k-1)}}{(m-i-l+k-1)!} \\
[\gamma\{(g+j+h+i+p1-n)I_{(g+j+h+i+p1-n)}-(g+j+h+i+p1+n+2)I_{(g+j+h+i+p1+n+2)}\}-\gamma^{-1}\{ \\
(g+j+h+i+p1-n+1)I_{(g+j+h+i+p1-n+1)}-(g+j+h+i+p1+n+3)I_{(g+j+h+i+p1+n+3)}\}]- \\
[2(\mu_1+\mu_2)\{\lambda p(\mu_2-\mu_1)+\mu_1\mu_2\}][\{(g+j+h+i+p1-n+1)I_{(g+j+h+i+p1-n+1)}- \\
(g+j+h+i+p1+n+1)I_{(g+j+h+i+p1+n+1)}\}-\gamma^{-1}\{(g+j+h+i+p1+n+2)I_{(g+j+h+i+p1+n+2)}- \\
(g+j+h+i+p1-n+2)I_{(g+j+h+i+p1-n+2)}\}](2\sqrt{\lambda(\mu_2+\mu_1)}u)^{-1}e^{-(\lambda+\xi)(w-u)}du]e^{-(\lambda+\beta-\xi)w}] \\
e^{-(\mu_1+\mu_2+\xi)(t-w)}dw-\sum_{j=0}^{\infty}\sum_{l=0}^{j+1}\sum_{k=0}^{\infty}\sum_{i=0}^{N+2}(-1)^{l+i}\binom{j+l}{l}\gamma^{-2(j+k+i+1)}[\lambda^2\{p(\mu_2+\mu_1)-\mu_2\}]^{l+i}\{(2(j+k+i) \\
-N-2)I_{[2(j+k+i)-N-2]}t^{-1}e^{-(\lambda+\mu_1+\mu_2+\xi)t}\}-[\xi\lambda^2\{(\lambda+\xi+\mu_1)^2+\lambda\mu_1\mu_2(\lambda+\xi+\mu_2)+\lambda^2\{p(\mu_2+\mu_1) \\
-\mu_2\}\}]\int_0^t\frac{(t-w)}{1!}[\{\sum_{i=0}^{N-2}\gamma^{-2(i+1)}(2i-N+3)I_{(2i-N+3)}\}(2\sqrt{\lambda(\mu_2+\mu_1)}w).w^{-1}]e^{-(\lambda+\mu_1+\mu_2+\xi)w} \\
-\lambda^{-1}\int_0^w\frac{(w-u)}{1!}[\{\sum_{i=0}^{n-1}\gamma^{-2(n-i)}(2i-n+1)I_{(2i-n+1)}-\sum_{i=0}^{n-2}\gamma^{-2(n-i-1)}(2i-n+2)I_{(2i-n+2)}\}(2\sqrt{\lambda(\mu_2+\mu_1)}u) \\
.u^{-1}.e^{-(\lambda+\mu_1+\mu_2+\xi)u}]e^{-(\lambda+\mu_2+\xi)(w-u)}du]e^{-(\lambda+\beta-\xi)w}]e^{-(\lambda+\mu_1+\xi)(t-w)}dw+[\xi\{(2\mu_2+\mu_1+\xi)(\mu_2+2\mu_1+\xi) \\
(\mu_2+\mu_1)-(\mu_2+\mu_1)\mu_2^{\,2}(1-p)+(\mu_2+\mu_1)\mu_2\mu_1-(\mu_2+\mu_1)\mu_1^{\,2}p\}]\sum_{j=0}^{\infty}\sum_{l=0}^{j+1}\sum_{k=0}^{\infty}\sum_{h=0}^{l}\sum_{r=0}^{\infty}\sum_{n=0}^{N}(-1)^{l+h}\binom{m+j}{j} \\
\binom{j+l}{l}\binom{l}{h}\gamma^{-2(j+k+r)}\int_0^t\frac{(t-w)^{(k-l+h-1)}}{(k-l+h-1)!}[\{I_{(g+j+h-l-n)}-I_{(g+j+h-l+n)}\}-\gamma^{-1}\{I_{(g+j+h-l-n+1)}-I_{(g+j+h-l+n+1)}\}] \\
(2\sqrt{\lambda(\mu_2+\mu_1)}w)e^{-(\lambda+\mu_1+\mu_2+\xi)(t-w)}+\xi\lambda^{-1}\int_0^t[\{\sum_{i=0}^{N-1}\gamma^{-2i+1}(2i-N+2)I_{(2i-N+2)}\}(2\sqrt{\lambda(\mu_2+\mu_1)}w)w^{-1} \\
e^{-(\lambda+\mu_1+\mu_2+\xi)w}+[(\mu_2+\mu_1)\lambda p]\int_0^w[\sum_{i=0}^{n-1}\gamma^{-2(n-i)}(2i-n+1)I_{(2i-n+1)}](2\sqrt{\lambda(\mu_2+\mu_1)}u)u^{-1}.e^{-(\lambda+\mu_1+\mu_2+\xi)u}] \\
e^{-(\lambda+\mu_1+\xi)(w-u)}du]e^{-(\lambda+\beta-\xi)w}]e^{-(\lambda+\mu_2+\xi)(t-w)}dw]
\end{array}\right]
$$

$$
Q_{m,00}(t)=\xi\left[\begin{array}{l}
\frac{\gamma^{2n}}{\lambda}[2(\mu_1+\mu_2)\gamma^{-1}][\sum_{j=0}^{\infty}\sum_{l=0}^{j+i+1}\sum_{k=0}^{\infty}\sum_{h=0}^{l}\sum_{r=0}^{\infty}\sum_{i=0}^{m-1}\sum_{q=0}^{j-l}\sum_{p1=0}^{h}\sum_{n=0}^{N}(-1)^{i+l+p1+q}\binom{m-1}{i}\binom{m+j}{j}\binom{j+i+1}{l} \\
\binom{l}{h}\binom{j+i+k}{k}\binom{h}{p1}\binom{j+l+r+q}{q}\lambda^{m-i-l+p1}[2(\mu_1+\mu_2)\{\lambda p(\mu_2-\mu_1)+\mu_1\mu_2\}]^{m-i-l-q+p1} \\
\gamma^{-(2m+j-i-q+1)}\int_0^t\frac{(t-w)^{(m-i-l+p1+q-1)}}{(m-i-l+p1+q-1)!}[\{I_{(g+j+h+i+p1+1)}-I_{(g+j+h+i+p1+2)}\}-\gamma^{-4} \\
\{I_{(g+j+h+i+p1+q+k)}-I_{(g+j+h+i+p1+q+k+2)}\}+\{I_{(g+j+h+i+p1+k+q+r+1)}-I_{(g+j+h+i+p1+k+q+r+3)}\} \\
](2\sqrt{\lambda(\mu_2+\mu_1)}w)e^{-(\lambda+\beta-\xi)w}]e^{-(\mu_1+\mu_2+\xi)(t-w)}dw+[\sum_{j=0}^{\infty}\sum_{l=0}^{j+i+2}\sum_{k=0}^{\infty}\sum_{h=0}^{l}\sum_{p1=0}^{h}\sum_{i=0}^{m-1}\sum_{n=0}^{N}(-1)^{i+l+h}\binom{m-1}{i}\binom{m+j}{j} \\
\binom{j+i+2}{l}\binom{l}{h}\binom{j+i+k+1}{k}\lambda^{m-i-l+p1}\gamma^{-(2m+j-i-l-2n-1)}\int_0^t\frac{(t-w)^{(m-i-l+p1-1)}}{(m-i-l+p1-1)!}\left[\int_0^w\frac{(w-u)^{(m-i-l+k-1)}}{(m-i-l+k-1)!}\right. \\
[\gamma\{(g+j+h+i+p1-n)I_{(g+j+h+i+p1-n)}-(g+j+h+i+p1+n+2)I_{(g+j+h+i+p1+n+2)}\}-\gamma^{-1}\{ \\
(g+j+h+i+p1-n+1)I_{(g+j+h+i+p1-n+1)}-(g+j+h+i+p1+n+3)I_{(g+j+h+i+p1+n+3)}\}]- \\
[2(\mu_1+\mu_2)\{\lambda p(\mu_2-\mu_1)+\mu_1\mu_2\}][\{(g+j+h+i+p1-n+1)I_{(g+j+h+i+p1-n+1)}- \\
(g+j+h+i+p1+n+1)I_{(g+j+h+i+p1+n+1)}\}-\gamma^{-1}\{(g+j+h+i+p1+n+2)I_{(g+j+h+i+p1+n+2)}- \\
(g+j+h+i+p1-n+2)I_{(g+j+h+i+p1-n+2)}\}](2\sqrt{\lambda(\mu_2+\mu_1)}u)^{-1}e^{-(\lambda+\xi)(w-u)}du]e^{-(\lambda+\beta-\xi)w}] \\
e^{-(\mu_1+\mu_2+\xi)(t-w)}dw-\sum_{j=0}^{\infty}\sum_{l=0}^{j+1}\sum_{k=0}^{\infty}\sum_{i=0}^{N+2}(-1)^{l+i}\binom{j+l}{l}\gamma^{-2(j+k+i+1)}[\lambda^2\{p(\mu_2+\mu_1)-\mu_2\}]^{l+i}\{(2(j+k+i) \\
-N-2)I_{[2(j+k+i)-N-2]}t^{-1}e^{-(\lambda+\mu_1+\mu_2+\xi)t}\}-[\xi\lambda^2\{(\lambda+\xi+\mu_1)^2+\lambda\mu_1\mu_2(\lambda+\xi+\mu_2)+\lambda^2\{p(\mu_2+\mu_1) \\
-\mu_2\}\}]\int_0^t\frac{(t-w)}{1!}[\{\sum_{i=0}^{N-2}\gamma^{-2(i+1)}(2i-N+3)I_{(2i-N+3)}\}(2\sqrt{\lambda(\mu_2+\mu_1)}w).w^{-1}]e^{-(\lambda+\mu_1+\mu_2+\xi)w} \\
-\lambda^{-1}\int_0^w\frac{(w-u)}{1!}[\{\sum_{i=0}^{n-1}\gamma^{-2(n-i)}(2i-n+1)I_{(2i-n+1)}-\sum_{i=0}^{n-2}\gamma^{-2(n-i-1)}(2i-n+2)I_{(2i-n+2)}\}(2\sqrt{\lambda(\mu_2+\mu_1)}u) \\
.u^{-1}.e^{-(\lambda+\mu_1+\mu_2+\xi)u}]e^{-(\lambda+\mu_2+\xi)(w-u)}du]e^{-(\lambda+\beta-\xi)w}]e^{-(\lambda+\mu_1+\xi)(t-w)}dw+[\xi\{(2\mu_2+\mu_1+\xi)(\mu_2+2\mu_1+\xi) \\
(\mu_2+\mu_1)-(\mu_2+\mu_1)\mu_2{}^2(1-p)+(\mu_2+\mu_1)\mu_2\mu_1-(\mu_2+\mu_1)\mu_1{}^2p\}]\sum_{j=0}^{\infty}\sum_{l=0}^{j+1}\sum_{k=0}^{\infty}\sum_{h=0}^{l}\sum_{r=0}^{\infty}\sum_{n=0}^{N}(-1)^{l+h} \\
\binom{m+j}{j}\binom{j+l}{l}\binom{l}{h}\gamma^{-2(j+k+r)}\int_0^t\frac{(t-w)^{(k-l+h-1)}}{(k-l+h-1)!}[\{I_{(g+j+h-l-n)}-I_{(g+j+h-l+n)}\}-\gamma^{-1}\{I_{(g+j+h-l-n+1)}- \\
I_{(g+j+h-l+n+1)}\}](2\sqrt{\lambda(\mu_2+\mu_1)}w)e^{-(\lambda+\mu_1+\mu_2+\xi)(t-w)}+\xi\lambda^{-1}\int_0^t[\{\sum_{i=0}^{N-1}\gamma^{-2i+1}(2i-N+2)I_{(2i-N+2)}\} \\
(2\sqrt{\lambda(\mu_2+\mu_1)}w)w^{-1}e^{-(\lambda+\mu_1+\mu_2+\xi)w}+[(\mu_2+\mu_1)\lambda p]\int_0^w[\sum_{i=0}^{n-1}\gamma^{-2(n-i)}(2i-n+1)I_{(2i-n+1)}] \\
(2\sqrt{\lambda(\mu_2+\mu_1)}u)u^{-1}.e^{-(\lambda+\mu_1+\mu_2+\xi)u}]e^{-(\lambda+\mu_1+\xi)(w-u)}du]e^{-(\lambda+\beta-\xi)w}]e^{-(\lambda+\mu_2+\xi)(t-w)}dw]
\end{array}\right]
$$

$$
\begin{aligned}
-\xi \Bigg[& [2(\mu_1+\mu_2)+\mu_1\mu_2]\sum_{j=0}^{\infty}\sum_{l=0}^{j+1}\sum_{k=0}^{\infty}\sum_{h=0}^{l}\sum_{r=0}^{\infty}\sum_{i=0}^{m-1}(-1)^{i+l+h}\binom{m+j}{j}\binom{j+1}{l}\binom{l}{h}\binom{j+l+r}{r} \\
& \binom{m-1}{i}\Big[2^{-(j+1)}\gamma^{-2(j+k+r-l)}\int_0^t \frac{(t-w)^{(m-i-l+k+h-1)}}{(m-i-l+k+h-1)!}\Big[[\{I_{(g+j+h+i-l-n-1)}-I_{(g+j+h+i-l+n+1)}\} \\
& -2\lambda(\mu_1+\mu_2)\gamma^{-4}\{I_{(g+j+h+i-l-n)}-I_{(g+j+h+i-l+n)}\}](2\sqrt{\lambda(\mu_1+\mu_2)}w)+[(\lambda+\xi)(\lambda+\xi+ \\
& \mu_1)+\lambda(\mu_1+\mu_2)][\int_0^w \frac{(w-u)^{(k+r-l-1)}}{(k+r-l-1)!}\{I_{(g+j+h+i-l-n)}-I_{(g+j+h+i-l+n)}\}(2\sqrt{\lambda(\mu_1+\mu_2)}u)] \\
& e^{-(\lambda+\xi+\mu_1)u}\Big]e^{-(\lambda+\beta-\xi)w}]e^{-(\mu_1+\mu_2+\xi)(t-w)}dw\Big]-[(\lambda+\xi+\mu_1)(\lambda+\xi+\mu_2)+\mu_1\mu_2] \\
& \Big[\sum_{j=0}^{\infty}\sum_{l=0}^{j+1}\sum_{k=0}^{\infty}\sum_{h=0}^{l}\sum_{r=0}^{\infty}\sum_{i=0}^{m-1}(-1)^{i+l+h}\binom{m+j}{j}\binom{j+l}{l}\binom{l}{h}\binom{j+l+r}{r}\binom{m-1}{i}\gamma^{-(i+k+l+6)}(\mu_1+\mu_2)^{l+h} \\
& \lambda^{m-i-l+h}2^{m-i-l+h+r}\int_0^t \frac{(t-w)^{(m-i-l+k-1)}}{(m-i-l+k-1)!}[\{I_{(g+j+h+i-l-n+1)}-I_{(g+j+h+i-l+n+1)}\}(2\sqrt{\lambda(\mu_1+\mu_2)}w)] \\
& e^{-(\lambda+\beta-\xi)w}]e^{-(\mu_1+\mu_2+\xi)(t-w)}dw-[\xi(\lambda+\xi+\mu_1)(\lambda+\xi+\mu_2)+\mu_1\mu_2]\sum_{j=0}^{\infty}\sum_{l=0}^{j+1}\sum_{k=0}^{\infty}\sum_{i=0}^{N-1}(-1)^{l}\binom{j+1}{l}\binom{j+k}{k} \\
& \gamma^{-2(j+k+i+1)}(\mu_1+\mu_2)^{l+i}(2(j+k+i)-N+1)I_{[2(j+k+i)-N+1]}t^{-1}e^{-(\lambda+\mu_1+\mu_2+\xi)t}+[3\{(\lambda+\xi+\mu_1) \\
& (\lambda+\xi+\mu_2)\}+\lambda\mu_1\mu_2(\lambda+\xi+\mu_2)+\xi\lambda(1-p)+\mu_1(1-p)(\lambda+\xi)\mu_1\mu_2] \\
& \sum_{j=0}^{\infty}\sum_{l=0}^{j+1}\sum_{k=0}^{\infty}\sum_{h=0}^{l}\sum_{r=0}^{\infty}(-1)^{l+h}\binom{j+1}{l}\binom{j+k}{k}\gamma^{-2(j+k)}\int_0^t \frac{(t-w)^{(k+h-l-1)}}{(k+h-l-1)!}[\{I_{(g+j+h-l-n)}-I_{(g+j+h-l+n)}\} \\
& -\gamma^{-1}\{I_{(g+j+h-l-n+1)}-I_{(g+j+h-l+n+1)}\}](2\sqrt{\lambda(\mu_1+\mu_2)}w)e^{-(\lambda+\beta-\xi)w}]e^{-(\lambda+\xi)w}dw+\xi[\lambda\xi(1-p)(\lambda+\xi+\mu_1) \\
& +(\mu_1+\mu_2)\mu_1\mu_2(1-p)\lambda+2\mu_2(\lambda+\xi)\mu_1]\Bigg[\int_0^t \frac{(t-w)^2}{2!}[\{\sum_{i=0}^{N-2}\gamma^{-2(i+1)}(2i-N+2)I_{(2i-N+2)}\} \\
& -\{\sum_{i=0}^{N-1}\gamma^{-2i}(2i-N+1)I_{(2i-N+1)}\}](2\sqrt{\lambda(\mu_1+\mu_2)}w)w^{-1}e^{-(\lambda+\mu_1+\xi)w}\Bigg]e^{-(\lambda+\beta-\xi)w}]e^{-(\lambda+\mu_1+\mu_2+\xi)(t-w)}dw+ \\
& [\lambda p(\mu_2-\mu_1)+\mu_1(\mu_2-1)+(\mu_1-1)\mu_2(1-p)+\mu_1(\lambda+\mu_1+\xi)^2+\mu_2(\lambda+\mu_2+\xi)^2] \\
& [\{3\int_0^t \frac{(t-w)}{1!}e^{-(\lambda+\xi+\mu_1)(t-w)}+4\int_0^t \frac{(t-w)^2}{2!}e^{-(\lambda+\xi+\mu_2)(t-w)}\}dw][\sum_{i=0}^{n-2}\gamma^{-2(n-i-1)}(2i-n+2)I_{(2i-n+2)}] \Bigg]
\end{aligned}
$$

(51)

Where $I_\nu(\alpha t)\equiv I_\nu, \alpha = 2\sqrt{\lambda(\mu_1+\mu_2)}$

4. NUMBER OF TIMES THE SYSTEM REACHES ITS CAPACITY IN TIME t

Putting y=1 in (26) and on simplification, we have

$$\bar{H}(x,1;s) = \frac{A_1}{s}\left[1+\lambda(x-1)P_{N-1}(x,s)\right] \qquad (52)$$

Using (28) in (52) and setting x=0, we have

$$\bar{P}_{0,.}(s) = \frac{A_1}{s}\left[1-\frac{Z_{15}}{Z_{16}}\right] \qquad (53)$$

Where

$$Z_{15} = \begin{bmatrix} [(s+\xi)\{A_1-(s+\mu_1+\lambda+\xi)\}-A_2(\mu_1+\mu_2)][V(1)\{\gamma^{-2}(B_1+\lambda B_2)-B_2(\mu_1+\mu_2)\} \\ -\lambda\gamma^{-2}B_2V(2)]+s[[(s+\xi)(s+\mu_1+\lambda+\xi)+A_2\lambda][\gamma^{-2}V(1)\{B_1-\lambda\gamma^{-2}B_2+B_2(\mu_1+\mu_2)\} \\ -V(2)B_2(\mu_1+\mu_2)]+A_2(\mu_1+\mu_2)[-V(1)\{B_1+B_2(\mu_1+\mu_2)\}-V(2)\{B_1-\lambda B_2\}-V(3)\lambda B_2]] \end{bmatrix}$$

$$Z_{16} = \xi\begin{bmatrix} \lambda A_1[(\mu_1+\mu_2)B_2V(N+1)-\lambda\gamma^{-6}B_2V(N-2)-\gamma^{-2}V(N)\{B_1+B_2(\mu_1+\mu_2)\} \\ +\gamma^{-4}V(N-1)\{B_1+\lambda B_2\}]-[\{\lambda A_2+(s+\mu_1+\lambda+\xi)\}\{\lambda\gamma^{-4}B_2V(1)+B_2(\mu_1+\mu_2)V(2) \\ -\gamma^{-2}V(1)\{B_1+B_2(\mu_1+\mu_2)\}\}+\{A_1-A_2(\mu_1+\mu_2)-(s+\mu_1+\lambda+\xi)\}\{B_2(\mu_1+\mu_2)V(1) \\ +\lambda\gamma^{-2}B_2V(2)-\gamma^{-2}\{B_1+\lambda B_2\}\}]] \end{bmatrix}$$

Using (28) in (52), differentiating m times w.r.t. x and dividing both sides by m! And setting x=0, we have

$$\bar{P}_{m,.}(s) = \left[\frac{Z_{17}}{Z_{18}}\right] \qquad (54)$$

Where

$$Z_{17} = \begin{bmatrix} [\gamma^{-2}V(1)\{B_1-\lambda\gamma^{-2}B_2+B_2(\mu_1+\mu_2)\}-B_2(\mu_1+\mu_2)V(2)][B_2(\mu_1+\mu_2)V(N+2)- \\ 2\lambda(s+\mu_1+\mu_2+\lambda+\xi)\{\gamma^{-2}[V(N-1)\gamma^{-4}B_2+V(N+1)]\}+B_2(\mu_1+\mu_2)+\gamma^{-4}V(N) \\ \{B_1+\lambda B_2\}][V(1)\{B_1+B_2(\mu_1+\mu_2)\}A_1-A_2(\mu_1+\mu_2)-(s+\mu_1+\lambda+\xi)\{B_2(\mu_1+\mu_2)V(1) \\ +\lambda\gamma^{-2}B_2V(2)+\lambda A_2(s+\mu_1+\lambda+\xi)\}+\lambda A_1[\lambda B_2V(N-2)-\gamma^{-2}V(N)\{B_1+B_2(\mu_1+\mu_2)\}] \\ +A_1A_2B_1\gamma^{-2}\{V(N+1)-V(N)\}+\lambda^2A_1B_2(\mu_1+\mu_2)\{V(N+2)-V(N+1)\}+\gamma^{-4}\{B_1+\lambda B_2\} \\ \{V(N+1)-V(N-2)\}]^{(m-1)} \end{bmatrix}$$

$$Z_{18} = \begin{bmatrix} [s\lambda\xi(s+\mu_1+\mu_2+\xi)^m s^{-(m+1)}(\lambda\xi)^{m+1}][2\lambda A_1[(\mu_1+\mu_2)B_2V(N+1)-\lambda\gamma^{-6}V(N-2)B_2]- \\ \gamma^{-2}V(N)\{B_1+(\mu_1+\mu_2)B_2\}-\gamma^{-4}V(N-1)\{B_1+\lambda B_2\}-\{A_1-A_2(\mu_1+\mu_2)-(s+\lambda+\mu_1+\xi)\} \\ \{(\mu_1+\mu_2)B_2V(1)+\lambda B_2\gamma^{-2}V(2)-\gamma^{-2}\{B_1+\lambda B_2\}\}]-[\lambda A_2+(s+\lambda+\mu_1+\xi)] \\ [\gamma^{-4}\lambda B_2V(1)+B_2V(2)(\mu_1+\mu_2)-\gamma^{-2}V(1)\{B_1+(\mu_1+\mu_2)B_2\}]]^{m+1} \end{bmatrix}$$

Expanding the right hand sides of (53) and (54), we have

$$\bar{P}_{0,}(s)=\begin{bmatrix} \sum_{j=0}^{\infty}\sum_{l=0}^{j+1}\sum_{k=0}^{\infty}\sum_{h=0}^{l}\sum_{i=0}^{j}\sum_{r=0}^{\infty}(-1)^{i+l+h}\binom{j+1}{l}\binom{l}{h}\binom{j+l+r}{r}\binom{j}{i}\binom{j+k-l}{k}\lambda^{-(j+l-h+i)}\gamma^{-2(j+k+h-l+r+1)}s^{-2} \\ \left[(\lambda+\xi+\mu_1)(\lambda+\xi+\mu_2)+\xi\lambda^2(\mu_2-\mu_1)p-\lambda\mu_2^2(1-p)\right][\{R^{(g+j+k+l-1)}-R^{(g+j+k+l+1)}\} \\ -\gamma^{-1}\{R^{(g+j+i+h+k+1)}-R^{(g+j+i+h+k+2)}\}+(2\lambda)^{i+j+k}\gamma^{-4}\{R^{(g+j+k+l-1)}-R^{(g+j+k+l-2)}\}]-\sum_{j=0}^{\infty}\sum_{l=0}^{j}\sum_{k=0}^{\infty}s^{-2} \\ \sum_{i=0}^{j}\sum_{r=0}^{\infty}(-1)^{i+l}\binom{j+l}{l}\binom{j+k-1}{k}\binom{j+r}{r}\gamma^{-(j+2k-l+i+2)}[\lambda(\lambda+\xi+\mu_1)(\lambda+\xi+\mu_2)(\mu_1+\mu_2) \\ -\lambda\mu_1^2(1-p)+\lambda p(\lambda+\xi+\mu_2)][\{R^{(g+j+l+k+N)}-R^{(g+j+l+k+N+1)}\}-\gamma^2\{R^{(g+j+l+k+1)}-R^{(g+j+l+k+2)}\}] \\ -[\lambda p(\mu_1+\mu_2)]\gamma^{-3}\{R^{(g+j-1)}-R^{(g+j+1)}\}]-\xi(\mu_1+\mu_2)(s+\lambda+\xi)^{-(k-l-1)}[\{R^{(g+j+r+l+1)}-R^{(g+j+r+l+2)}\}] \end{bmatrix} \quad (55)$$

$$\bar{P}_{m,}(s)=\begin{bmatrix} \sum_{j=0}^{\infty}\sum_{l=0}^{j+1}\sum_{k=0}^{\infty}\sum_{h=0}^{l}\sum_{r=0}^{\infty}\sum_{i=0}^{m-1}\sum_{q=0}^{j-k}\sum_{p1=0}^{m+i-1}(-1)^{i+l+p1+q+h}\binom{m-1}{i}\binom{m+j}{j}\binom{j+1}{l}\binom{l}{h}\binom{j+k}{k}\binom{m+i-1}{p1}\binom{j-k}{q} \\ \binom{j+i+l+r}{r}\lambda^{m-i-l+p1-q-1}2^{m-i-l+p1+q}[(\mu_1+\mu_2)\lambda p]^{l+p1}\gamma^{-(2m+j-i-q-1)}(s+\mu_1+\mu_2+\xi)^{-(m-i-l+p1-1)} \\ (s+\lambda+\xi)^{-(m-i-k-l-1)}[\{R^{(g+j+h+i+p1+N)}-R^{(g+j+h+i+p1+N+1)}\}-\gamma\{R^{(g+j+h+i+p1+N+2)}-R^{(g+j+h+i+p1+N+3)}\}] \\ +\sum_{j=0}^{\infty}\sum_{l=0}^{j+1}\sum_{k=0}^{\infty}\sum_{i=0}^{m-1}(-1)^{i+l}\binom{m-1}{i}\binom{m+j}{j}\binom{j+i}{l}\binom{j+i+k}{k}\lambda^{m-i-k-1}\gamma^{-(2m+j-i-N-2)}[2\lambda(\lambda+\xi+\mu_1) \\ (\lambda+\xi+\mu_2)(\mu_1+\mu_2)](s+\mu_1+\mu_2+\xi)^{-(m-i+l+k)}[\{R^{(g+j+i+N)}-R^{(g+j+i+N+2)}\}-\gamma\{R^{(g+j+i+N+1)} \\ -R^{(g+j+i+N+3)}\}]-\sum_{j=0}^{\infty}\sum_{l=0}^{j+1}\sum_{k=0}^{\infty}\sum_{i=0}^{N+2}(-1)^{l+i}\binom{j+l}{l}\gamma^{-2(j+k+i+1)}[\lambda^2\{p(\mu_2+\mu_1)-\mu_2\}]^{l+i}\{(2\lambda)^{2(j+k+i)-N-2} \\ [(s+\lambda+\mu_1+\mu_2+\xi)+\nu]^{-[2(j+k+i)-N-2]}\}+[\lambda(\lambda+\xi+\mu_1)(\lambda+\xi+\mu_2)(\mu_1+\mu_2)-\lambda\mu_1^2(1-p) \\ +\lambda p^2(\mu_1+\mu_2)][\{\sum_{i=0}^{N-2}\gamma^{-2(i+1)}(2\lambda)^{2i-N+3}[(s+\lambda+\mu_1+\mu_2+\xi)+\nu]^{-(2i-N+3)}\}-\{\sum_{i=0}^{n-1}\gamma^{2(n-i)}(2\lambda)^{(2i-n+1)} \\ [(s+\lambda+\mu_1+\mu_2+\xi)+\nu]^{-(2i-n+1)}\}] \end{bmatrix} \quad (56)$$

Taking the inverse transform of (55) and (56), we have

$$P_{0,.}(t)=\left[\begin{aligned}&\sum_{j=0}^{\infty}\sum_{l=0}^{j+1}\sum_{k=0}^{\infty}\sum_{h=0}^{l}\sum_{i=0}^{j}\sum_{r=0}^{\infty}(-1)^{i+l+h}\binom{j+1}{l}\binom{l}{h}\binom{j+l+r}{r}\binom{j}{i}\binom{j+k-l}{k}\lambda^{-(j+l-h+i)}\gamma^{-2(j+k+h-l+r+1)}\\&\left[(\lambda+\xi+\mu_1)(\lambda+\xi+\mu_2)+\xi\lambda^2(\mu_2-\mu_1)p-\lambda\mu_2^{\,2}(1-p)\right]\int_0^t\frac{(t-w)}{1!}[\{(g+j+k+l)(g+j+k+l-1)\\&(\sqrt{\lambda(\mu_1+\mu_2)}w)^{-1}I_{(g+j+k+l)}-2I_{(g+j+k+l+1)}\}-\gamma^{-1}\{(g+j+i+h+k+1)^2(\sqrt{\lambda(\mu_1+\mu_2)}w)^{-1}I_{(g+j+i+h+k+1)}\\&-2I_{(g+j+i+h+k+2)}\}+(2\lambda)^{i+j+k}\gamma^{-4}\{(g+j+k+l-3)(g+j+k+l-1)(\sqrt{\lambda(\mu_1+\mu_2)}w)^{-1}I_{(g+j+k+l-3)}\\&-2I_{(g+j+k+l-2)}\}]w^{-1}e^{-(\lambda+\mu_1+\mu_2+\xi)(t-w)}dw-\sum_{j=0}^{\infty}\sum_{l=0}^{j}\sum_{k=0}^{\infty}\sum_{i=0}^{j}\sum_{r=0}^{\infty}(-1)^{i+l}\lambda^{-(j+i+1)}\binom{j+l}{l}\binom{j+k-1}{k}\binom{j+r}{r}\\&\gamma^{-(j+2k-l+i+2)}[\lambda(\lambda+\xi+\mu_1)(\lambda+\xi+\mu_2)(\mu_1+\mu_2)-\lambda\mu_1^{\,2}(1-p)+\lambda p(\lambda+\xi+\mu_2)]\int_0^t\frac{(t-w)}{1!}[[\{I_{(g+j+l+k+N)}\\&-I_{(g+j+l+k+N+1)}\}-\gamma^2\{I_{(g+j+l+k+1)}-I_{(g+j+l+k+2)}\}-[\lambda p(\mu_1+\mu_2)]\gamma^{-3}\{I_{(g+j-1)}-I_{(g+j+1)}\}](2\sqrt{\lambda(\mu_1+\mu_2)}w)\\&-\xi(\mu_1+\mu_2)\int_0^w\frac{(w-u)^{(k-l-2)}}{(k-l-2)!}[\{I_{(g+j+r+l+1)}-I_{(g+j+r+l+2)}\}(2\sqrt{\lambda(\mu_1+\mu_2)}u)e^{-(\lambda+\xi)u}du]e^{-(\lambda+\mu_1+\mu_2+\xi)(t-w)}dw]\end{aligned}\right]\tag{57}$$

$$P_{m,.}(t)=\left[\begin{aligned}&\sum_{j=0}^{\infty}\sum_{l=0}^{j+1}\sum_{k=0}^{\infty}\sum_{h=0}^{l}\sum_{r=0}^{\infty}\sum_{i=0}^{m-1}\sum_{q=0}^{j-k}\sum_{p1=0}^{m+i-1}(-1)^{i+l+p1+q+h}\binom{m-1}{i}\binom{m+j}{j}\binom{j+1}{l}\binom{l}{h}\binom{j+k}{k}\binom{m+i-1}{p1}\binom{j-k}{q}\\&\binom{j+i+l+r}{r}\lambda^{m-i-l+p1-q-1}2^{m-i-l+p1+q}[(\mu_1+\mu_2)\lambda p]^{l+p1}\gamma^{-(2m+j-i-q-1)}\int_0^t\frac{(t-w)^{(m-i-l+p1-2)}}{(m-i-l+p1-2)!}[\\&\int_0^w\frac{(w-u)^{(m-i-l-k-2)}}{(m-i-l-k-2)!}[\{I_{(g+j+h+i+p1+N)}-I_{(g+j+h+i+p1+N+1)}\}-\gamma\{I_{(g+j+h+i+p1+N+2)}-I_{(g+j+h+i+p1+N+3)}\}]\\&(2\sqrt{\lambda(\mu_1+\mu_2)}u)e^{-(\lambda+\xi)u}du]e^{-(\mu_1+\mu_2+\xi)(t-w)}dw]+\sum_{j=0}^{\infty}\sum_{l=0}^{j+1}\sum_{k=0}^{\infty}\sum_{i=0}^{m-1}(-1)^{i+l}\binom{m-1}{i}\binom{m+j}{j}\binom{j+i}{l}\\&\binom{j+i+k}{k}\lambda^{m-i-k-1}\gamma^{-(2m+j-i-N-2)}[2\lambda(\lambda+\xi+\mu_1)(\lambda+\xi+\mu_2)(\mu_1+\mu_2)]\int_0^t\frac{(t-w)^{(m-i+l+k-1)}}{(m-i+l+k-1)!}\\&[\{I_{(g+j+i+N)}-I_{(g+j+i+N+2)}\}-\gamma\{I_{(g+j+i+N+1)}-I_{(g+j+i+N+3)}\}](2\sqrt{\lambda(\mu_1+\mu_2)}w)e^{-(\mu_1+\mu_2+\xi)(t-w)}dw\\&-\sum_{j=0}^{\infty}\sum_{l=0}^{j+1}\sum_{k=0}^{\infty}\sum_{i=0}^{N+2}(-1)^{l+i}\binom{j+l}{l}\gamma^{-2(j+k+i+1)}[\lambda^2\{p(\mu_2+\mu_1)-\mu_2\}]^{l+i}\{(2(j+k+i)-N-2)\\&I_{[2(j+k+i)-N-2]}t^{-1}e^{-(\lambda+\mu_1+\mu_2+\xi)t}\}+[\lambda(\lambda+\xi+\mu_1)(\lambda+\xi+\mu_2)(\mu_1+\mu_2)-\lambda\mu_1^{\,2}(1-p)+\lambda p^2(\mu_1+\mu_2)]\\&[\{\sum_{i=0}^{N-2}\gamma^{-2(i+1)}(2i-N+3)I_{(2i-N+3)}\}-\{\sum_{i=0}^{n-1}\gamma^{2(n-i)}(2i-n+1)I_{(2i-n+1)}\}]t^{-1}e^{-(\lambda+\mu_1+\mu_2+\xi)t}\end{aligned}\right]\tag{58}$$

5. MEASURE OF EFFECTIVENESS

Two measures of effectiveness of immediate interest are the expectation $\mu_{X(t)}(N)$ and the variance $\sigma^2_{X(t)}(N)$ of the distribution of the number of times the system reaches its capacity in time interval (0, t].

5(a). Expectation $\{\mu_{X(t)}(N)\}$

$$\bar{\mu}_{X(s)}(N) = \frac{d}{dx}\left[\bar{H}(x,1;s)\right]_{x=1} \tag{59}$$

Expanding the right hand side of (59), we have

$$\bar{\mu}_{X(s)}(N) = \left[\begin{array}{l} [\lambda p(\mu_2-\mu_1)+\lambda(\lambda+\mu_1+\xi)(\lambda+\mu_2+\xi)+\mu_1\lambda-\mu_2\lambda(1-p)]\sum_{j=0}^{\infty}\sum_{i=0}^{k}\sum_{l=0}^{j+k+1}\sum_{p1=0}^{l}\sum_{k=0}^{\infty}(-1)^{i+l} \\ \binom{j+k}{j}\binom{j+k+l}{l}\binom{l}{p1}\binom{k+1}{i}\lambda^{-(2j+i+l)}\gamma^{-2(j-i-k)+k(N+1)}(s+\mu_1+\mu_2+\xi)^{-(j+i-l+p1)} \\ \left[\begin{array}{l}\{R^{-(H+1)}-R^{-(H-1)}\}+\gamma^{-2}\{R^{-(j+i+k+l+N+2)}-R^{-(j+i+k+l+N+3)}\}+[\mu_2\lambda(1-p)(\lambda+\mu_1+\xi) \\ +\lambda^2 p(\lambda+\xi)(\mu_2+\mu_1)](s+\lambda+\xi)^{-(j-i+l+p1)}\{R^{-(H+1)}-R^{-(H-1)}\}\end{array}\right]+ \\ [\lambda p(\mu_2+\mu_1)(\lambda+\mu_1+\xi)+(\mu_2-\mu_1)(\lambda+\mu_2+\xi)][\sum_{i=0}^{N-2}\gamma^{-2(i+1)}(2\lambda)^{2i-N+2} \\ [(s+\lambda+\mu_1+\mu_2+\xi)+v]^{-(2i-N+2)}-\lambda^{-1}\sum_{i=0}^{N-1}\gamma^{-2(N-i)}(2\lambda)^{2i-N+1}[(s+\lambda+\mu_1+\mu_2+\xi)+v]^{-(2i-N+1)}] \end{array}\right] \tag{60}$$

Where H= (j+i+k+l+1) (N+1)

Taking the inverse transform of (60), we have

$$\mu_{X(t)}(N) = \left[\begin{array}{l} [\lambda p(\mu_2-\mu_1)+\lambda(\lambda+\mu_1+\xi)(\lambda+\mu_2+\xi)+\mu_1\lambda-\mu_2\lambda(1-p)]\sum_{j=0}^{\infty}\sum_{i=0}^{k}\sum_{l=0}^{j+k+1}\sum_{p1=0}^{l}\sum_{k=0}^{\infty}(-1)^{i+l} \\ \binom{j+k}{j}\binom{j+k+l}{l}\binom{l}{p1}\binom{k+1}{i}\lambda^{-(2j+i+l)}\gamma^{-2(j-i-k)+k(N+1)}\int_0^t\frac{(t-w)^{j+i-l+p1-1}}{(j+i-l+p1-1)!}[\int_0^w\frac{(w-u)}{1!} \\ [[\{I_{(H+1)}-I_{(H-1)}\}+\gamma^{-2}\{I_{(j+i+k+l+N+2)}-I_{(j+i+k+l+N+3)}\}](2\sqrt{\lambda(\mu_2+\mu_1)}u)+[\mu_2\lambda(1-p) \\ (\lambda+\mu_1+\xi)+\lambda^2 p(\lambda+\xi)(\mu_2+\mu_1)]\int_0^u\frac{(u-v)^{(j-i+l+p1-1)}}{(j-i+l+p1-1)!}[\{I_{(H+1)}-I_{(H-1)}\}(2\sqrt{\lambda(\mu_2+\mu_1)}v) \\ ]e^{-(\lambda+\xi)(u-v)}dv]e^{-(\lambda+\mu_1+\mu_2+\xi)(w-u)}du]e^{-(\mu_1+\mu_2+\xi)(t-w)}dw+[\lambda p(\mu_2+\mu_1)(\lambda+\mu_1+\xi)+ \\ (\mu_2-\mu_1)(\lambda+\mu_2+\xi)][\{\sum_{i=0}^{N-2}\gamma^{-2(i+1)}(2i-N+2)I_{2i-N+2}\}-\lambda^{-1}\{\sum_{i=0}^{N-1}\gamma^{-2(N-i)}(2i-N+1)I_{2i-N+1}\} \\ ]t^{-1}e^{-(\mu_1+\mu_2+\xi)t} \end{array}\right] \tag{61}$$

5(b). Variance $\{\sigma^2_{X(t)}(N)\}$

$$\sigma^2_{X(t)}(N) = K(t) + \mu_{X(t)}(N)[1 - \mu_{X(t)}(N)] \tag{62}$$

Where

$$K(t) = \sum_{m=2}^{\infty} m(m-1)P_{m,.}(t)$$

Therefore, to find the value of variance, it suffices to calculate the first term on right hand side of (62) since expectation has already been obtained in (61).Now Laplace transform of K(t) is

$$\bar{K}(s) = \frac{d^2}{dx^2}\left[\bar{H}(x,1;s)\right]_{x=1} \tag{63}$$

Expanding the right hand side of (63), we have

$$\bar{K}(s) = \left[\begin{array}{l}
[\lambda(\lambda+\xi)\{(\mu_2-\mu_1)(1-p)+(\mu_1+\mu_2)\lambda^2-\mu_1\mu_2(\lambda+\xi)+(\lambda+\xi+\mu_1)(\lambda+\xi+\mu_2)] \\
\sum_{j=0}^{\infty}\sum_{l=0}^{j+1}\sum_{k=0}^{i}\sum_{h=0}^{l}\sum_{i=0}^{\infty}\sum_{p1=0}^{\infty}(-1)^{k+l+h}\binom{i+1}{i}\binom{j+i+1}{j}\binom{j+1}{l}\binom{l}{h}\binom{i}{k}\binom{j+l+p1}{p1}\lambda^{-(j+i+k+l)} \\
\gamma^{-(2(k+i+l)+p1(N+1)+1)}(s+\mu_1+\mu_2+\xi)^{-(j+i+k+p1)}[\{R^{-(v-1)}-R^{-(v+1)}\}-\gamma^{-1}\{R^{-(j+i+k+l+N+1)}- \\
R^{-(j+i+k+l+N+2)}\}+(s+\lambda+\xi)^{-(j+i+2)}[\{R^{-(N+v+1)}-R^{-(v+N+2)}\}]]+2(\mu_1+\mu_2)(s+\mu_1+\mu_2+\xi) \\
(s+\lambda+\xi)^3\sum_{j=0}^{\infty}\sum_{l=0}^{i}\sum_{k=0}^{\infty}(-1)^l\binom{i}{h}\gamma^{2N}[\{R^{-\{2(k+1)+1\}\{N+1\}}-R^{-\{2(k+1)-1\}\{N+1\}}\}]+[(\lambda+\xi)(\lambda+\xi+\mu_1) \\
(\lambda+\xi+\mu_2)-\lambda\mu_1 p(\lambda+\xi+\mu_2)-\lambda\mu_2(1-p)(\lambda+\xi+\mu_1)][\{\sum_{i=0}^{N-2}\gamma^{-2(i+1)}(2\lambda)^{2i-N+2} \\
[(s+\lambda+\mu_1+\mu_2+\xi)+v]^{-(2i-N+2)}\}-\sum_{i=0}^{N-1}\gamma^{-2(N-i)}(2\lambda)^{2i-N+1}[(s+\lambda+\mu_1+\mu_2+\xi)+v]^{-(2i-N+1)}]
\end{array}\right] \tag{64}$$

Where v = [{2(j+k+l+1) +i+p1} (N+1)]

Taking the Inverse Laplace transform of (64), we have

$$K(t)=\left[\begin{array}{l}
[\lambda(\lambda+\xi)\{(\mu_2-\mu_1)(1-p)+(\mu_1+\mu_2)\lambda^2-\mu_1\mu_2(\lambda+\xi)+(\lambda+\xi+\mu_1)(\lambda+\xi+\mu_2)] \\
\sum_{j=0}^{\infty}\sum_{l=0}^{j+1}\sum_{k=0}^{i}\sum_{h=0}^{l}\sum_{i=0}^{\infty}\sum_{p1=0}^{\infty}(-1)^{k+l+h}\binom{i+1}{i}\binom{j+i+1}{j}\binom{j+1}{l}\binom{l}{h}\binom{i}{k}\binom{j+l+p1}{p1}\lambda^{-(j+i+k+l)} \\
\gamma^{-(2(k+i+l)+p1(N+1)+1)}\int_0^t\frac{(t-w)^{(j+i+k+p1-1)}}{(j+i+k+p1-1)!}[[\{I_{(v-1)}-I_{(v+1)}\}-\gamma^{-1}\{I_{(j+i+k+l+N+1)}- \\
I_{(j+i+k+l+N+2)}\}](2\sqrt{\lambda(\mu_1+\mu_2)}w)+\int_0^w\frac{(w-u)^{(j+i+1)}}{(j+i+1)!}[\{I_{(N+v+1)}-I_{(v+N+2)}\}(2\sqrt{\lambda(\mu_1+\mu_2)}u) \\
]e^{-(\lambda+\xi)u}du]e^{-(\lambda+\mu_1+\mu_2+\xi)w}]e^{-(\mu_1+\mu_2+\xi)(t-w)}dw+[2(\mu_1+\mu_2)]\sum_{j=0}^{\infty}\sum_{l=0}^{i}\sum_{k=0}^{\infty}(-1)^l\binom{i}{h}\gamma^{2N}\int_0^t(t-w) \\
[\{2+(\mu_1+\mu_2+\xi)(t-w)][[\{\{2(k+1)+1\}\{N+1\}]][\{2(k+1)+1\}\{N+1\}-1] \\
(\sqrt{\lambda(\mu_1+\mu_2)}w)^{-1}I_{\{2(k+1)+1\}\{N+1\}}-2I_{\{2(k+1)-1\}\{N+1\}}\}]w^{-1}e^{-(\lambda+\mu_1+\mu_2+\xi)w}]e^{-(\lambda+\xi)(t-w)}dw+ \\
[(\lambda+\xi)(\lambda+\xi+\mu_1)(\lambda+\xi+\mu_2)-\lambda\mu_1p(\lambda+\xi+\mu_2)-\lambda\mu_2(1-p)(\lambda+\xi+\mu_1)] \\
[\{\sum_{i=0}^{N-2}\gamma^{-2(i+1)}(2i-N+2)I_{(2i-N+2)}\}-\sum_{i=0}^{N-1}\gamma^{-2(N-i)}(2i-N+1)I_{(2i-N+1)}]t^{-1}e^{-(\lambda+\mu_1+\mu_2+\xi)t}
\end{array}\right] \qquad (65)$$

REFERENCES

1. B.K.Kumar and S.P.Madheswari, "Transient solution of an M/M/2 queue with Heterogeneou servers subject to Catastrophes," information and management science, Vol. 18, No. 1, pp.63-80, 2007.
2. D.V.Widder (1946), "The Laplace Transform", Princeton Mathematical Series.
3. Di Crescenzo, A; Giorno, V and Ricciardi, L.M. (2003). "On the M/M/1 queue with Catastrophes and its continuous Approximation," Queueing systems, 43,329-347.
4. Erdelyi, A, W.Magnus, F.Oberhettinger and F.G.Tricomi (1954),"Table of Integral Transform",Vol.1, McGraw-Hill Book Company, New York.
5. G.N.Watson, Sc.D, F.R.S. (1959), "A Treatise on the Theory of Bessel Function", 2nd Edition, Cambridge Uni. Press.
6. S.P.Madheswari and B.K.Kumar, "An M/M/2 queueing system with heterogeneous Servers and multiple vacations," Mathematical and Computing modeling, 13(41), Pp.1415-1429, 2005.